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QUADRATIC FORMS AND THEIR CLASSIFICATION BY MEANS OF INVARIANT-FACTORS

by

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PREFACE.

IN writing this tract, the chief difficulty has been to compress the material so as not to entirely outrun the prescribed limits of space.

The theory is developed in an order which may seem unusual to readers already acquainted with other methods of treatment; but my object has been to obtain a fairly complete account in the minimum of space. If the methods of Weierstrass or Darboux had been adopted, a long and rather tedious discussion would have been needed for certain determinantal theorems (Arts. 24, 25), before the real problem of reduction could have been attacked. Further, the singular case would then have required an entirely separate discussion, of which the only satisfactory account[1] is both involved and laborious.

Both of these objections are avoided by the method used here, which is due in substance to Kronecker. And, in addition, the method lends itself to geometrical explanations (see Arts. 1, 13, 17 and Appendix) and is well adapted for the actual reduction of numerical examples, when once the roots of the fundamental determinant are known (see Arts. 2, 16, 19, 22). I hope that a frequent appeal to geometry may serve to make the algebra more easily understood.

The omission of any account of Weierstrass's and Darboux's methods would be a serious blot, if the tract were intended to be

[1] **This is Kronecker's discussion in the third paper quoted in Art. 38: the account in Muth's *Elementartheiler* (pp. 93—117) is shorter, because restricted to *bilinear* forms. But the extension to quadratic forms then depends on a theorem due to Frobenius (Muth, pp. 127, 128); and an explanation of this theorem would have required an addition of some 20 pages more.**

exhaustive rather than suggestive; in particular Darboux's treatment of the case of unequal roots[1] must always be regarded as a model of algebraical elegance. But accounts of these methods are already available to a certain extent (for references, see Art. 38); and consequently an exposition is less necessary here.

I have devoted Chapter V to an exhibition of some applications of the theory; these may serve to convince the reader of its utility; and a glance at the table given in Art. 23 will shew that families containing more than four variables could not be exhaustively classified without the aid of invariant-factors. Indeed, even the case of four variables was not fully worked out (in spite of the assistance derived from space-intuition) until Sylvester took the first step in the general theory by classifying the contacts of quadric surfaces (see Arts. 11, 18).

In conclusion, my best thanks are due to the editors for giving me the opportunity of writing this tract; and to Mr Leathem in particular for reading the manuscript and proofs. His care has enabled me to detect and remove many difficulties and ambiguities; but it is only too likely that others remain to be found after publication. The addition of the Appendix is due to Mr Leathem's suggestion.

The printing of the University Press stands in no need of praise from me; but I must thank the officials for their excellent reproductions of my drawings, and for their careful superintendence of the press-work in general.

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GALWAY,
May, 1906.

[1] For English readers, this discussion is accessible in Burnside and Panton's *Theory of Equations*, Art. 210, Ex. 17, and Scott's *Theory of Determinants*, Chap. XI, § 10.

CONTENTS.

CHAPTER I.

INTRODUCTORY.

1. IT is well known that two quadratic expressions in a variable x, can usually be expressed as the sum or difference of multiples of the same two squares; that is, numbers p, q, r, s, α, β can generally be found* to satisfy the identities

$$\left.\begin{aligned} S = ax^2 + 2bx + c = p(x-\alpha)^2 + q(x-\beta)^2 \\ S' = a'x^2 + 2b'x + c' = r(x-\alpha)^2 + s(x-\beta)^2 \end{aligned}\right\} \text{............(1)},$$

when a, b, c, a', b', c' are given.

Suppose for the moment that the coefficients of S, S' are all real, and let us represent the values of x geometrically by distances measured from a fixed origin on a fixed straight line. Then the pairs of points given by $\lambda S - S' = 0$, form an involution on the line; and the two points α, β are the *double points* (or *foci*) of the involution. And our statement simply amounts to saying that the involution determined by the two pairs of points $S = 0$, $S' = 0$ has usually a pair of double points.

It is a familiar fact that the involution determined by two given pairs of real distinct points on a straight line belongs to one of two classes; the *elliptic* (when the two segments on the line overlap) and the *hyperbolic* (when the two segments do not overlap). In the former case, the double points are imaginary; in the latter, they are real†.

The case when a point belonging to one pair coincides with a point belonging to the other pair may be called *parabolic*: this case is not of special interest from the geometrical point of view, since the involution is degenerate and the two double points are coincident. However, it

* For an example of the work in an actual case, see Art. 2.

† An algebraic investigation will be found in Chrystal's *Algebra*, vol. I, pp. 464—6. It should be noticed incidentally that if either pair of given points is imaginary, the double points will be real (compare Ex. 3, Art. 3).

exhibits some features of interest on the algebraic side and we shall now proceed to illustrate the parabolic and hyperbolic cases by numerical examples.

2. Consider first the determination of the double points for the two quadratics

$$S = 3x^2 - 34x + 91, \qquad S' = 5x^2 - 42x + 93.$$

For this purpose, find λ so as to make $\lambda S - S'$ a square* as regards x; the equation giving λ is

$$(3\lambda - 5)(91\lambda - 93) - (17\lambda - 21)^2 = 0,$$

or
$$-4(4\lambda^2 + 5\lambda - 6) = 0.$$

Thus λ is either $\frac{3}{4}$ or -2; and on inserting these values we find

$$-3S + 4S' = 11(x-3)^2, \quad 2S + S' = 11(x-5)^2,$$

so that
$$S = -(x-3)^2 + 4(x-5)^2,$$
$$S' = 2(x-3)^2 + 3(x-5)^2.$$

Hence the double points are given by $x = 3$, $x = 5$, and the involution is hyperbolic.

The foregoing method is on the whole the simplest in any specific case; and it serves as a convenient introduction to the methods to be used in the sequel.

We consider next a parabolic case in which the double points coincide; namely

$$S = 3x^2 - 34x + 91, \qquad S' = 5x^2 - 42x + 49.$$

Then $\lambda S - S'$ is a square if

$$(3\lambda - 5)(91\lambda - 49) - (17\lambda - 21)^2 = 0,$$

or if
$$-4(4\lambda^2 - 28\lambda + 49) = 0.$$

Hence $\lambda = \frac{7}{2}$, which gives

$$7S - 2S' = 11(x-7)^2,$$

and there is no other value of λ, which makes $\lambda S - S'$ a square as regards x.

It is therefore impossible to apply the same method as in the last case; but it does not follow immediately that S and S' cannot be reduced to the type (1) of Art. 1. As a matter of fact, such a reduction is not possible, for we find that

$$S = (x-7)(3x-13), \quad S' = (x-7)(5x-7),$$

* In the notation of the last article, λ will be either r/p or s/q.

so that S and S' have a common factor, as might have been expected from the geometrical interpretation. But, if the two expressions

$$p(x-\alpha)^2+q(x-\beta)^2, \qquad r(x-\alpha)^2+s(x-\beta)^2$$

have a common factor, either q and s must both be zero, or else

$$\sqrt{(p/q)}=\pm\sqrt{(r/s)};$$

and, whichever alternative is adopted, we find

$$ps-qr=0,$$

so that the one expression is simply a numerical multiple of the other. Consequently the two expressions

$$S=(x-7)(3x-13), \qquad S'=(x-7)(5x-7)$$

are *not* reducible to the type (1) of Art. 1; for they have a common factor, and yet S' is not a numerical multiple of S.

3. From the two examples worked out in the last article it is clear that when S, S' are given three possible cases may occur :—

(i) If $|\lambda S-S'|$ has two unequal factors, the reduction of S, S' can then be carried out by the method given at the beginning of Art. 2.

(ii) If $|\lambda S-S'|$ has a squared factor $(\lambda-k)^2$, but $(\lambda-k)$ does not divide *all* the expressions $(a\lambda-a')$, $(b\lambda-b')$, $(c\lambda-c')$, then S, S' have a common factor.

(iii) If $|\lambda S-S'|$ has a squared factor $(\lambda-k)^2$, but $\lambda-k$ is a factor of *all* the expressions $(a\lambda-a')$, $(b\lambda-b')$, $(c\lambda-c')$, then $S'=kS$ identically.

Here $|\lambda S-S'|$ denotes the determinant

$$\begin{vmatrix} a\lambda-a', & b\lambda-b' \\ b\lambda-b', & c\lambda-c' \end{vmatrix}$$

which, when equated to zero, expresses the condition that $\lambda S-S'$ may be a square as regards x.

Of these statements, those in (i) and (iii) are obvious without further proof; but the statement in (ii) requires investigation. Under the given circumstances we may write

$$kS-S'=a_1x^2+2b_1x+c_1,$$

where $a_1c_1-b_1^2=0$ and not *all* of a_1, b_1, c_1 can be zero. Then if $\lambda-k=\mu$, we have

$$\lambda S-S'=(a\mu+a_1)x^2+2(b\mu+b_1)x+(c\mu+c_1),$$

so that $$|\lambda S - S'| = (a\mu + a_1)(c\mu + c_1) - (b\mu + b_1)^2$$
$$= (ac - b^2)\mu^2 + (ac_1 - 2bb_1 + a_1 c)\mu.$$

But this determinant is divisible by μ^2, and therefore
$$ac_1 - 2bb_1 + a_1 c = 0.$$

If $a_1 \neq 0$, we can write $kS - S' = a_1(x-\alpha)^2$, so that $b_1 = -a_1\alpha$, $c_1 = a_1\alpha^2$; and then the last condition gives
$$a\alpha^2 + 2b\alpha + c = 0.$$

Thus* $(x - \alpha)$ is a factor of S, and therefore also of S'.

On the other hand, if $a_1 = 0$, we must have $b_1 = 0$, since $a_1c_1 - b_1^2 = 0$; and accordingly $c_1 \neq 0$. But $ac_1 - 2bb_1 + a_1c = 0$; so that here $a = 0$. Now $a' = ak - a_1 = 0$, and therefore S, S' are not proper quadratics; but if we put them in the homogeneous forms
$$S = 2bxy + cy^2, \quad S' = 2b'xy + c'y^2,$$
they evidently have the common factor y.

In the foregoing work it has been tacitly supposed that S is not itself a square; when this occurs it will be convenient to work with the determinant $|S - \lambda S'|$, which will obviously have λ as one factor. With the necessary modifications, the results already obtained still hold good (compare Ex. 5 below). It may of course happen that both S and S' are squares; they will then belong to the type (1) of Art. 1, without further reduction.

For a classification by means of invariant-factors, the reader may refer to the end of Art. 11.

EXAMPLES.

1. Reduce:

(i) $S = 2x^2 - 10x + 17$, $S' = 4x^2 - 26x + 49$. [*Math. Trip.* Pt I., 1891.]

(ii) $S = 12x^2 + 32x + 21$, $S' = 8x^2 - 6x - 27$.

[The results may be found in the *Quarterly Journal of Mathematics*, vol. XXXIII, 1901, p. 93.]

* The reader may find it instructive to observe that if β approaches α as a limit, while p, q tend to infinity in such a way that $p + q$ and $p(\beta - \alpha)$ have finite limits (p', q'), then the form $p(x-\alpha)^2 + q(x-\beta)^2$ has the limit
$$p'(x-\alpha)^2 + 2q'(x-\alpha) = (x-\alpha)[p'(x-\alpha) + 2q'].$$
Thus the limit of the two forms given in (1) Art. 1 leads to the case (ii) discussed in the text.

2. Prove that the double points of the involution $\lambda S - S' = 0$ are the roots of the quadratic in x

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ 1, & -x, & x^2 \end{vmatrix} = 0.$$

3. The coefficients of S and S' are real, prove that if $|\lambda S - S'| = 0$ has complex roots in λ, the type (1) of Art. 1 leads to the forms

$$S = l[(x-\gamma)^2 - \delta^2] + 2m(x-\gamma), \quad S' = l'[(x-\gamma)^2 - \delta^2] + 2m'(x-\gamma),$$

where l, m, l', m', γ, δ are real.

Deduce that the equation $\lambda S - S' = 0$ has real roots for all real values of λ.

[*Math. Trip.* Pt I., 1900.]

4. If x_1, x_2 are the roots of $S=0$ and y_1, y_2 of $S'=0$, prove that

$$\frac{(a-x_1)(a-y_1)}{(a-x_2)(a-y_2)} = \frac{(\beta-x_1)(\beta-y_1)}{(\beta-x_2)(\beta-y_2)},$$

and that the equation is still valid when y_1, y_2 are interchanged.

[*Math. Trip.* Pt I., 1894.]

5. If $S = ax^2 + 2bx + c$, and $S' = r(x-a)^2$, we have $S = p(x-a)^2 + q(x-\beta)^2$, where

$$\beta = -\frac{ba+c}{aa+b}, \quad p = \frac{ac-b^2}{aa^2+2ba+c}, \quad q = \frac{(aa+b)^2}{aa^2+2ba+c}.$$

This transformation is obviously always possible unless $(x-a)$ is a factor of S.

6. In the notation of equation (1) Art. 1

$$p = (ac-b^2)/(aa^2+2ba+c).$$

[*St John's Coll. Exam.* 1892.]

7. If x_1, x_2 are connected by the relation

$$ax_1x_2 + b(x_1+x_2) + c = 0,$$

then
$$\frac{a'x_1^2+2b'x_1+c'}{ax_1^2+2bx_1+c} + \frac{a'x_2^2+2b'x_2+c'}{ax_2^2+2bx_2+c} = \frac{r}{p} + \frac{s}{q} = \frac{ac'+a'c-2bb'}{ac-b^2}.$$

[Write $y = (x-a)/(x-\beta)$, so that $py_1y_2 + q = 0$.]

A number of special numerical examples for reduction will be found on pp. 23—4 of Greenhill's *Chapter on the Integral Calculus*, in the form of integrals of the type $\int dx/S\sqrt{S'}$.

CHAPTER II.

THEOREMS ON A SINGLE QUADRATIC FORM.

4. Preliminary Lemma. *If the determinant*

$$\begin{vmatrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ a_{21}, & a_{22}, & \dots, & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & \dots, & a_{nn} \end{vmatrix}$$

is zero, multipliers $l_1, l_2, \dots, l_n$ can be found such that

$$l_1 X_1 + l_2 X_2 + \dots + l_n X_n = 0$$

identically (that is, for all *values of $x_1, x_2, \dots, x_n$); where*

$$X_r = a_{r1}x_1 + a_{r2}x_2 + \dots + a_{rn}x_n ;$$

and a_{rs} is not necessarily equal to a_{sr}.

For this identity requires the set of linear equations to be satisfied :—

$$a_{11}l_1 + a_{21}l_2 + a_{31}l_3 + \dots + a_{n1}l_n = 0,$$

$$a_{12}l_1 + a_{22}l_2 + a_{32}l_3 + \dots + a_{n2}l_n = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{1n}l_1 + a_{2n}l_2 + a_{3n}l_3 + \dots + a_{nn}l_n = 0.$$

As the reader probably knows, the vanishing of the determinant $|a_{rs}|$ is the condition that these equations may hold simultaneously; in fact $l_1, l_2, \dots, l_n$ may then be taken to be proportional to the first minors of the elements in any column of the determinant, assuming that the first minors do not *all* vanish.

When the first minors are all zero, but at least one second minor is not zero, there will be *two independent* linear relations amongst the

X's. For the purpose of simplifying the argument, we shall take $n=5$ and suppose that the minor

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix}$$

is not zero*. Then consider the value of the determinant

$$\Delta = \begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & X_1 \\ a_{21}, & a_{22}, & a_{23}, & X_2 \\ a_{31}, & a_{32}, & a_{33}, & X_3 \\ a_{41}, & a_{42}, & a_{43}, & X_4 \end{vmatrix}.$$

When the X's are written out in full it is seen that Δ is equal to

$$x_4 \begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & a_{14} \\ a_{21}, & a_{22}, & a_{23}, & a_{24} \\ a_{31}, & a_{32}, & a_{33}, & a_{34} \\ a_{41}, & a_{42}, & a_{43}, & a_{44} \end{vmatrix} + x_5 \begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & a_{15} \\ a_{21}, & a_{22}, & a_{23}, & a_{25} \\ a_{31}, & a_{32}, & a_{33}, & a_{35} \\ a_{41}, & a_{42}, & a_{43}, & a_{45} \end{vmatrix}$$

and the determinants multiplying x_4 and x_5 are each zero, because they are first minors. Thus Δ is zero; and so, on expanding Δ in terms of its last column, we have a linear relation between X_1, X_2, X_3, X_4; and the coefficients in this relation are not *all* zero, because X_4 is multiplied by the non-zero second minor†.

In like manner we can obtain a linear relation between X_1, X_2, X_3, X_5, which must be independent of the first linear relation, unless X_4 and X_5 are identically equal; in that special case, the second linear relation is simply $X_4 = X_5$.

Of course, there will be generally more than two relations amongst the five X's, but as a matter of fact the other relations are deducible from these two; a proof of this statement will not be given here, as the theorem is not needed in what follows.

The reader should now have no difficulty in convincing himself of the truth of the following theorem:—

If all the minors of the $(k-1)$th *order are zero, but at least one minor of the* kth *order is not zero, there will be* k *independent linear relations amongst the* X's.

The converse theorem is also true but will not be needed here.

* If the a's are supposed to be related so that $a_{12}=a_{21}$, $a_{13}=a_{31}$, etc. (or in any other way which involves a symmetry about the principal diagonal), it might be necessary to allow the rows selected to be different from the columns.

† Otherwise the relation might simply reduce to

$$0 . X_1 + 0 . X_2 + 0 . X_3 + 0 . X_4 = 0.$$

5. If the determinant of a quadratic form in n variables is zero, it can be expressed in terms of $(n-1)$ variables at most.

Let the original variables be $x_1, x_2, \ldots, x_n$ and let the quadratic form be

$$A = a_{11}x_1^2 + a_{22}x_2^2 + \ldots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + \ldots$$

so that a_{rr} denotes the coefficient of x_r^2, while that of $2x_rx_s$ is $a_{rs} = a_{sr}$. Then

$$\tfrac{1}{2}\frac{\partial A}{\partial x_1} = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n,$$

$$\tfrac{1}{2}\frac{\partial A}{\partial x_2} = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tfrac{1}{2}\frac{\partial A}{\partial x_n} = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n.$$

Hence by Art. 4 there is at least one linear relation with *constant* coefficients amongst the partial differential coefficients of A, because

$$|A| = \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}, & a_{n2}, & \ldots, & a_{nn} \end{vmatrix} = 0.$$

We may suppose the suffixes arranged in such a way that the coefficient of $\partial A/\partial x_1$ in this relation is not zero; so that we may take the relation to be

$$\frac{\partial A}{\partial x_1} = c_2\frac{\partial A}{\partial x_2} + c_3\frac{\partial A}{\partial x_3} + \ldots + c_n\frac{\partial A}{\partial x_n},$$

where some or all of $c_2, c_3, \ldots, c_n$ may be zero.

Now introduce the new set of variables

$$y_1 = x_1, \quad y_2 = x_2 + c_2x_1, \quad y_3 = x_3 + c_3x_1, \quad \ldots, \quad y_n = x_n + c_nx_1,$$

and we see that

$$x_1 = y_1, \quad x_2 = y_2 - c_2y_1, \quad \ldots, \quad x_n = y_n - c_ny_1,$$

so that
$$\frac{\partial A}{\partial y_1} = \frac{\partial A}{\partial x_1} - c_2\frac{\partial A}{\partial x_2} - c_3\frac{\partial A}{\partial x_3} - \ldots - c_n\frac{\partial A}{\partial x_n} = 0,$$

and
$$\frac{\partial A}{\partial y_2} = \frac{\partial A}{\partial x_2}, \quad \ldots, \quad \frac{\partial A}{\partial y_n} = \frac{\partial A}{\partial x_n}.$$

From the fact that $\frac{\partial A}{\partial y_1} = 0$, it follows that A, *when expressed in terms of the y's, does not contain y_1; so that A depends only on the $(n-1)$ variables $y_2, y_3, \ldots, y_n$.*

It may, of course, happen that in consequence of some minors of $|A|$ being zero, there is more than one linear relation between the partial differential coefficients $\partial A/\partial x$; if there are k such relations, there will be $(k-1)$ relations between $\partial A/\partial y_2$, $\partial A/\partial y_3$, ..., $\partial A/\partial y_n$. Proceeding similarly, we can prove finally that A depends only on $(n-k)$ variables*.

Familiar special cases of the general theorem are the following:—

(i) $n=2$; *the quadratic form* $ax^2+2bxy+cy^2$ *is a perfect square if its determinant* $(ac-b^2)$ *is zero.* (Compare Arts. 1—3.)

(ii) $n=3$; *the conic*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

degenerates into a pair of lines if its determinant is zero; and the two lines coincide if the first minors are all zero.

(iii) $n=4$; *the quadric* $\Sigma a_{rs}x_r x_s=0$ $(r, s=1, 2, 3, 4)$, *degenerates into a cone if its determinant is zero; into a pair of planes if all the first minors are zero; into a pair of coincident planes if all the second minors are zero.*

It is convenient to have a name for the *least* number of independent variables in terms of which a quadratic form can be expressed; we shall adopt the term *rank* (German *Rang*) for this number.

6. We shall give next a method of transforming a single quadratic form A. Consider the determinant

$$K=\begin{vmatrix} a_{11}, & a_{12}, & \dots, & a_{1k}, & X_1 \\ a_{21}, & a_{22}, & \dots, & a_{2k}, & X_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{k1}, & a_{k2}, & \dots, & a_{kk}, & X_k \\ X_1, & X_2, & \dots, & X_k, & A \end{vmatrix},$$

where $X_1, X_2, \dots, X_k$ have the same values as in Art. 4; but now (as in Art. 5) $a_{rs}=a_{sr}$, and of course k is less than n.

If we multiply the first k columns of K by $x_1, x_2, \dots, x_k$ in order, and subtract from the last column; and then treat the rows in the same way, it will be seen that K no longer contains any of the variables $x_1, x_2, \dots, x_k$. Now these operations do not alter the value of the

* First stated explicitly by Sylvester, *Phil. Mag.* series 4, vol. I, 1851, p. 121; *Coll. Papers*, vol. I, p. 221. But the substance of the theorem is contained in an earlier paper of 1850 (*Coll. Papers*, vol. I, p. 145); this contains the further theorem that all the kth minors are zero if $\frac{1}{2}(k+1)(k+2)$ of them (properly chosen) are zero.

determinant K, so that K must be independent of $x_1, x_2, \ldots, x_k$; thus we may write

$$K = f(x_{k+1}, x_{k+2}, \ldots, x_n),$$

where the form of f is most quickly calculated by putting

$$x_1 = 0, \quad x_2 = 0, \quad \ldots, \quad x_k = 0$$

in the determinant K.

But if we expand K by the elements of the last column it is clear that K is also equal to

$$A \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1k} \\ a_{21}, & a_{22}, & \ldots, & a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}, & a_{k2}, & \ldots, & a_{kk} \end{vmatrix} + \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1k}, & X_1 \\ a_{21}, & a_{22}, & \ldots, & a_{2k}, & X_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1}, & a_{k2}, & \ldots, & a_{kk}, & X_k \\ X_1, & X_2, & \ldots, & X_k, & 0 \end{vmatrix}.$$

Thus, provided that the determinant

$$D_k = \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1k} \\ a_{21}, & a_{22}, & \ldots, & a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}, & a_{k2}, & \ldots, & a_{kk} \end{vmatrix}$$

is different from zero, we have a transformation of A given by

$$AD_k = f(x_{k+1}, x_{k+2}, \ldots, x_n) - \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1k}, & X_1 \\ a_{21}, & a_{22}, & \ldots, & a_{2k}, & X_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1}, & a_{k2}, & \ldots, & a_{kk}, & X_k \\ X_1, & X_2, & \ldots, & X_k, & 0 \end{vmatrix} \quad \ldots(1).$$

In many applications it is useful to further transform the determinant containing $X_1, X_2, \ldots, X_k$, by writing

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + \ldots + a_{1k}y_k &= X_1, \\ a_{21}y_1 + a_{22}y_2 + \ldots + a_{2k}y_k &= X_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{k1}y_1 + a_{k2}y_2 + \ldots + a_{kk}y_k &= X_k; \end{aligned}$$

then since the determinant D_k of the coefficients on the left-hand side has been supposed different from zero, we can express $y_1, y_2, \ldots, y_k$ as linear functions of $X_1, X_2, \ldots, X_k$; and it will be readily seen that the differences

$$y_1 - x_1, \quad y_2 - x_2, \quad \ldots, \quad y_k - x_k$$

depend only on $\qquad x_{k+1}, \quad x_{k+2}, \quad \ldots, \quad x_n.$

If we substitute for $X_1, X_2, \ldots, X_k$ in terms of $y_1, y_2, \ldots, y_k$, we see from (1) that

$$-\begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1k}, & X_1 \\ a_{21}, & a_{22}, & \ldots, & a_{2k}, & X_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1}, & a_{k2}, & \ldots, & a_{kk}, & X_k \\ X_1, & X_2, & \ldots, & X_k, & 0 \end{vmatrix} = D_k A',$$

where A' is obtained from A, by writing $y_1, y_2, \ldots, y_k$ in place of $x_1, x_2, \ldots, x_k$ and zero for the other variables. Hence

$$A = A' + f(x_{k+1}, x_{k+2}, \ldots, x_n)/D_k \quad \ldots\ldots\ldots\ldots\ldots(2).$$

We shall discuss, in Art. 7, various forms of this transformation which will be required in the sequel. The transformation can also be used to establish Thomson's and Bertrand's theorems in Analytical Dynamics; this application, although unnecessary for our future work, is sufficiently interesting to be included here.

It will be seen that the above transformation (1) may be written in the form

$$A = g(X_1, X_2, \ldots, X_k) + h(x_{k+1}, x_{k+2}, \ldots, x_n),$$

where g, h are quadratic forms; the important point to notice being that in this transformed form of A there are no product terms of the type $X_r x_s$.

Now let $\frac{1}{2}A$ represent the kinetic energy of a body or dynamical system started from rest by a set of impulses; the variables $x_1, x_2, \ldots, x_n$ denoting the generalised velocities (in the sense associated with Lagrange's dynamical equations). Then, since A must be *positive* for all conceivable distributions of velocity, it is clear that the quadratic forms g, h must possess the same property; for we can suppose

$$X_1 = 0, \quad X_2 = 0, \quad \ldots, \quad X_k = 0,$$

without imposing any relation* on the variables

$$x_{k+1}, \quad x_{k+2}, \quad \ldots, \quad x_n,$$

and consequently $\qquad h(x_{k+1}, x_{k+2}, \ldots, x_n)$

must be positive for all real non-zero values of its arguments. A similar argument applies to $g(X_1, X_2, \ldots, X_k)$.

* For the determinant D_k is supposed not to vanish, so that the conditions

$$X_1 = 0, \quad X_2 = 0, \quad \ldots, \quad X_k = 0$$

enable us to express $x_1, x_2, \ldots, x_k$ in terms of the remaining variables. As a matter of fact D_k cannot be zero; for if $D_k = 0$, we can find *non-zero* values of $x_1, x_2, \ldots, x_k$ to satisfy the conditions

$$X_1 = 0, \quad X_2 = 0, \quad \ldots, \quad X_k = 0, \quad x_{k+1} = 0, \quad x_{k+2} = 0, \quad \ldots, \quad x_n = 0;$$

and because $A = X_1 x_1 + X_2 x_2 + \ldots + X_n x_n$, these values would make A zero, contrary to the physical interpretation of A.

Now suppose the system started from rest in such a way that the velocity is prescribed at each point where an impulse is applied; then we may suppose $x_{k+1}, x_{k+2}, \ldots, x_n$ given. By Lagrange's equations for impulsive motion we have in the resulting motion

$$X_1 = 0, \quad X_2 = 0, \quad \ldots, \quad X_k = 0;$$

so that in the actual motion A is less than it would be in any other motion having the prescribed velocities*. This is *Thomson's (Lord Kelvin's) theorem.*

Closely associated with the last result is *Bertrand's theorem*; in this case we suppose some of the generalised impulses known, and contrast the actual motion with another motion, which is subject to additional constraints. It may be assumed that these further constraints are expressed by the equations

$$x_{k+1} = 0, \quad x_{k+2} = 0, \quad \ldots, \quad x_n = 0,$$

and the values of $X_1, X_2, \ldots, X_k$ will be known†. Then plainly the kinetic energy in the actual motion will exceed the kinetic energy in the constrained motion by

$$\tfrac{1}{2} h\,(x_{k+1}, x_{k+2}, \ldots, x_n),$$

which is positive; so that the actual motion has a *maximum* kinetic energy.

7. We now proceed to the special cases of the transformation of Art. 6, which will be of chief use to us; and it is to be remembered throughout this article that one of the variables is regarded as on a different footing from the rest; the special variable will be taken to be x_1. For geometrical interpretations, the reader may consult the Appendix.

(i) Suppose that a_{11} is not zero; then by taking $k = 1$ in Art. 6, equation (1), we have

$$a_{11} A = X_1^2 + f(x_2, x_3, \ldots, x_n).$$

We have thus isolated all the terms containing x_1, without modifying $x_2, \ldots, x_n$. Hence, if we write

$$y_1 = X_1/a_{11} = x_1 + (a_{12} x_2 + \ldots + a_{1n} x_n)/a_{11}$$

we find

$$A = a_{11} y_1^2 + f/a_{11} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

(ii) If a_{11} is zero, suppose that a_{12} is not zero‡, then take $k = 2$ in Art. 6, equation (1) and we find

$$a_{12}^2 A = 2a_{12} X_1 X_2 - a_{22} X_1^2 - f(x_3, x_4, \ldots, x_n).$$

* The difference between the two values of A is in fact $g(X_1, X_2, \ldots, X_k)$.

† Some of these may be zero, in case no impulse is applied to the corresponding coordinate.

‡ If x_1 is actually present in A, at least one of the coefficients $a_{12}, a_{13}, \ldots, a_{1n}$ must be different from zero; this one is supposed to be a_{12}.

Thus $$A = \frac{X_1}{a_{12}}\left(2X_2 - \frac{a_{22}}{a_{12}}X_1\right) - \frac{1}{a_{12}^2}f,$$

so write $$2y_1 = 2X_2 - a_{22}X_1/a_{12} = 2X_2 - a_{22}y_2,$$

$$y_2 = X_1/a_{12} = x_2 + (a_{13}x_3 + \dots + a_{1n}x_n)/a_{12},$$

and then $$A = 2y_1y_2 - f/a_{12}^2 \quad\dots\dots(2).$$

We have here isolated the terms in x_1 and x_2, without modifying $x_3, x_4, \dots, x_n$; and the modification of x_2 consists only in the addition of multiples of $x_3, x_4, \dots, x_n$.

(iii) Suppose that the first minor

$$D_{11} = \begin{vmatrix} a_{22}, & a_{23}, & \dots, & a_{2n} \\ a_{32}, & a_{33}, & \dots, & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{n2}, & a_{n3}, & \dots, & a_{nn} \end{vmatrix}$$

is not zero. Then the determinant

$$\begin{vmatrix} a_{22}, & a_{23}, & \dots, & a_{2n}, & X_2 \\ a_{32}, & a_{33}, & \dots, & a_{3n}, & X_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n2}, & a_{n3}, & \dots, & a_{nn}, & X_n \\ X_2, & X_3, & \dots, & X_n, & A \end{vmatrix}$$

is seen to be independent of $x_2, x_3, \dots, x_n$ by the method used in Art. 6 to evaluate K; and the same method shews that its value is Dx_1^2, where D denotes the determinant $|A|$.

Then, since D_{11} is not zero, we can find $y_2, y_3, \dots, y_n$ from the equations

$$a_{22}y_2 + a_{23}y_3 + \dots + a_{2n}y_n = X_2,$$
$$a_{32}y_2 + a_{33}y_3 + \dots + a_{3n}y_n = X_3,$$
$$\dots\dots\dots\dots$$
$$a_{n2}y_2 + a_{n3}y_3 + \dots + a_{nn}y_n = X_n,$$

just as in the second transformation of Art. 6. Thus the differences

$$(y_2 - x_2), (y_3 - x_3), \dots, (y_n - x_n)$$

are multiples only of x_1, and

$$A = A' + (D/D_{11})x_1^2 \quad\dots\dots(3)$$

by Art. 6, equation (2)*; where A' is obtained from A by writing zero for x_1, and $y_2, y_3, \dots, y_n$ for $x_2, x_3, \dots, x_n$ respectively.

* Here of course Dx_1^2 takes the place of the function $K = f(x_{k+1}, x_{k+2}, \dots, x_n)$.

By this transformation we reduce the terms containing x_1 to a single term, and the other variables are subjected to no other change than the addition of multiples of x_1.

(iv) If the first minor D_{11} is zero, while D is not zero*, there must be at least one element in the first row of D which gives a non-zero first minor: let us call this element a_{12}, and write

$$-D_{12} = \begin{vmatrix} a_{21}, & a_{23}, & a_{24}, & \dots, & a_{2n} \\ a_{31}, & a_{33}, & a_{34}, & \dots, & a_{3n} \\ a_{41}, & a_{43}, & a_{44}, & \dots, & a_{4n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n3}, & a_{n4}, & \dots, & a_{nn} \end{vmatrix}$$

so that D_{12} is not zero. We shall denote the minor of a_{21} by D_{21}, so that $D_{12} = D_{21}$.

By a familiar theorem† on determinants whose elements are first minors, we have

$$D \,.\, \begin{vmatrix} a_{33}, & a_{34}, & \dots, & a_{3n} \\ a_{43}, & a_{44}, & \dots, & a_{4n} \\ \dots & \dots & \dots & \dots \\ a_{n3}, & a_{n4}, & \dots, & a_{nn} \end{vmatrix} = \begin{vmatrix} D_{11}, & D_{12} \\ D_{21}, & D_{22} \end{vmatrix},$$

where D_{22} is the first minor of a_{22} in D. Thus, since $D_{11} = 0$, we have

$$\begin{vmatrix} a_{33}, & a_{34}, & \dots, & a_{3n} \\ a_{43}, & a_{44}, & \dots, & a_{4n} \\ \dots & \dots & \dots & \dots \\ a_{n3}, & a_{n4}, & \dots, & a_{nn} \end{vmatrix} = -D_{12}^2/D \quad \dots\dots\dots(4),$$

because $D_{12} = D_{21}$; and so the second minor in (4) is not zero.

Now, if we apply the method of Art. 6, it will be seen that

$$\begin{vmatrix} a_{33}, & a_{34}, & \dots, & a_{3n}, & X_3 \\ a_{43}, & a_{44}, & \dots, & a_{4n}, & X_4 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n3}, & a_{n4}, & \dots, & a_{nn}, & X_n \\ X_3, & X_4, & \dots, & X_n, & A \end{vmatrix} = D_{22}x_1^2 - 2D_{12}x_1x_2 + D_{11}x_2^2;$$

* If D is zero, the method of Art. 5 can be applied to express A in terms of fewer variables.

† Burnside and Panton's *Theory of Equations*, vol. II, § 146; or Scott's *Theory of Determinants*, Chap. V, § 6.

and since $D_{11} = 0$, it will be found on using equation (4), that

$$A = 2\frac{D}{D_{12}}x_1x_2 - \frac{DD_{22}}{D_{12}^2}x_1^2 + \frac{D}{D_{12}^2}\begin{vmatrix} a_{33}, & \dots, & a_{3n}, & X_3 \\ \dots & \dots & \dots & \dots \\ a_{n3}, & \dots, & a_{nn}, & X_n \\ X_3, & \dots, & X_n, & 0 \end{vmatrix}.$$

Again, $y_3, y_4, \dots, y_n$ can be found from the equations

$$a_{33}y_3 + a_{34}y_4 + \dots + a_{3n}y_n = X_3,$$
$$a_{43}y_3 + a_{44}y_4 + \dots + a_{4n}y_n = X_4,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{n3}y_3 + a_{n4}y_4 + \dots + a_{nn}y_n = X_n,$$

because the determinant of the left-hand side is simply the second minor of D which has been proved different from zero. Then each of the differences $(y_3 - x_3)$, $(y_4 - x_4)$, …, $(y_n - x_n)$ will be a linear function of x_1 and x_2.

After introducing the variables $y_3, \dots, y_n$, it will seen that as in Art. 6, equation (2)* we have

$$A = A' + 2(D/D_{12})\,x_1y_2 \quad \dots\dots\dots\dots\dots\dots(5),$$

where $y_2 = x_2 - \frac{1}{2}(D_{22}/D_{12})\,x_1$, and A' is obtained from A by writing zero for x_1 and x_2, and $y_3, y_4, \dots, y_n$ for $x_3, x_4, \dots, x_n$.

We have thus reduced the terms in x_1 to a single product, the other variables being subject to no other change than the addition of multiples of x_1 and x_2; in particular x_2 is only changed by the addition of a multiple of x_1.

8. Reduction of a single quadratic form to a sum of squares.

If we repeat the operations (i) and (ii) of Art. 7, we shall arrive at a reduced form in which each new variable appears once only; if the variable appears as a square, no further transformation need be applied to it. If two variables are multiplied together, forming a term such as $2ax_1x_2$, we can take $x_1 + x_2 = y_1$ and $x_1 - x_2 = y_2$ as new variables; the term then becomes $\frac{1}{2}a(y_1^2 - y_2^2)$.

In this way the form is reduced to a sum of r squares, where r is the rank of the form (Art. 5). Naturally the reduction is not unique, because we are free to collect the terms in any order we please.

* Compare p. 11; here $f(x_{k+1}, x_{k+2}, \dots, x_n)$ is represented by

$$D_{22}x_1^2 - 2D_{12}x_1x_2.$$

For instance, take

$$x^2+xy+y^2=(x+\tfrac{1}{2}y)^2+\tfrac{3}{4}y^2=\tfrac{3}{4}x^2+(y+\tfrac{1}{2}x)^2=\tfrac{3}{4}(x+y)^2+\tfrac{1}{4}(x-y)^2$$
$$=\frac{\tfrac{3}{4}(x-ay)^2+[(\tfrac{1}{2}+a)x+(1+\tfrac{1}{2}a)y]^2}{1+a+a^2},$$

where a may have *any* value which does not make the denominator zero.

A simple general formula for the transformation may be obtained as follows: write

$$A_k=\begin{vmatrix} a_{11}, & a_{12}, & \dots, & a_{1k}, & X_1 \\ a_{21}, & a_{22}, & \dots, & a_{2k}, & X_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{k1}, & a_{k2}, & \dots, & a_{kk}, & X_k \\ X_1, & X_2, & \dots, & X_k, & A \end{vmatrix}, \quad D_k=\begin{vmatrix} a_{11}, & a_{12}, & \dots, & a_{1k} \\ a_{21}, & a_{22}, & \dots, & a_{2k} \\ \dots & \dots & \dots & \dots \\ a_{k1}, & a_{k2}, & \dots, & a_{kk} \end{vmatrix},$$

and let ξ_k denote the value obtained when we replace the last row (or last column) of D_k by $X_1, X_2, \dots, X_k$. It is useful to note that ξ_k contains none of the variables $x_1, x_2, \dots, x_k$.

Then A_{k-1}, ξ_k, D_k are three first minors of A_k, and the complementary second minor of A_k is D_{k-1}. Hence

$$\begin{vmatrix} A_{k-1}, & \xi_k \\ \xi_k, & D_k \end{vmatrix} = D_{k-1}A_k,$$

which gives
$$\frac{A_{k-1}}{D_{k-1}}-\frac{A_k}{D_k}=\frac{\xi_k^2}{D_{k-1}D_k}$$

provided that neither D_k nor D_{k-1} is zero.

Further
$$A_1=\begin{vmatrix} a_{11}, & X_1 \\ X_1, & A \end{vmatrix}=D_1A-\xi_1^2$$

provided that we take $a_{11}=D_1$, $X_1=\xi_1$, in agreement with the general sequence; and A_n is easily seen to be identically zero. Thus we may write

$$A-\frac{A_1}{D_1}=\frac{\xi_1^2}{D_0D_1}, \qquad (\text{with } D_0=1)$$
$$\frac{A_1}{D_1}-\frac{A_2}{D_2}=\frac{\xi_2^2}{D_1D_2},$$
$$\dots\dots\dots\dots$$
$$\frac{A_{n-1}}{D_{n-1}}-0=\frac{\xi_n^2}{D_{n-1}D_n}.$$

Hence, on addition

$$A=\sum_{k=1}^{n}\frac{\xi_k^2}{D_{k-1}D_k} \quad\dots\dots\dots\dots(1),$$

assuming that $D_1, D_2, \dots, D_n$ are all different from zero.

If it happens that *all* the minors containing $(r+1)$ rows and columns are zero (though not *all* containing r rows and columns) it is easy to see that $A_r = 0$, so that the summation in equation (1) will extend only from $k=1$ to $k=r$; and r will be the rank of A (see p. 9).

Example 1. Take the form

$$A = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

then
$$D_0 = 1, \quad D_1 = a, \quad D_2 = ab - h^2 = C, \quad D_3 = \Delta,$$
$$\xi_1 = ax + hy + gz, \quad \xi_2 = Cy - Fz, \quad \xi_3 = \Delta z,$$

where the notation used is that familiar in connection with conics.

Thus
$$A = \frac{1}{a}(ax + hy + gz)^2 + \frac{1}{aC}(Cy - Fz)^2 + \frac{\Delta}{C}z^2,$$

assuming that a, C, Δ are not zero; this is capable of immediate verification, by applying the method (i) of Art. 7. Of course there are five other similar reductions, according to the order in which the variables are taken.

Another transformation may be noted (a special case of (1) Art. 6). With

$$X = ax + hy + gz, \quad Y = hx + by + fz,$$

we find
$$A = (aY^2 - 2hXY + bX^2 + \Delta z^2)/C.$$

Example 2 (*Math. Trip.* 1905). Take the form

$$A = 2\cosh\theta\,(x_1^2 + x_2^2 + \dots + x_n^2) + 2(x_1x_2 + x_2x_3 + \dots + x_{n-1}x_n).$$

Then it will be found that

$$D_{k+1} = 2\cosh\theta \,.\, D_k - D_{k-1},$$

so that, as
$$D_1 = 2\cosh\theta = \sinh 2\theta/\sinh\theta,$$

and
$$D_0 = 1 = \sinh\theta/\sinh\theta,$$

we get
$$D_k = \sinh(k+1)\theta/\sinh\theta.$$

Again we find

$$\xi_k = \begin{vmatrix} 2\gamma, & 1, & 0, & 0, & \dots, & 2\gamma x_1 + x_2 \\ 1, & 2\gamma, & 1, & 0, & \dots, & x_1 + 2\gamma x_2 + x_3 \\ 0, & 1, & 2\gamma, & 1, & \dots, & x_2 + 2\gamma x_3 + x_4 \\ 0, & 0, & 1, & 2\gamma, & \dots, & x_3 + 2\gamma x_4 + x_5 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

to k rows and columns, where γ is $\cosh\theta$. Multiply the columns in order by $x_1, x_2, \dots, x_{k-1}$, and subtract from the last column; it will then be seen that

$$\xi_k = D_{k-1}(2\gamma x_k + x_{k+1}) - x_k D_{k-2}$$
$$= D_k x_k + D_{k-1} x_{k+1}.$$

Or
$$\xi_k \sinh\theta = x_k \sinh(k+1)\theta + x_{k+1}\sinh k\theta.$$

Thus
$$A = \sum_{k=1}^{n} \frac{[x_k \sinh(k+1)\theta + x_{k+1}\sinh k\theta]^2}{\sinh(k+1)\theta \,.\, \sinh k\theta}, \text{ where } x_{n+1} = 0.$$

As a verification, note that the coefficient of x_k^2 in the last expression is

$$[\sinh (k-1)\theta + \sinh (k+1)\theta]/\sinh k\theta = 2\cosh\theta,$$

while the coefficient of $x_k x_{k+1}$ is obviously 2.

In case several of the determinants $D_1, D_2, \ldots, D_n$ are zero, it is not easy to give a general formula of so simple a character as (1) above; the best, from the point of view of elegance, is that of Darboux*. However, if D_k is zero while D_{k-1} and D_{k+1} are not zero, we can obtain an expression for the difference

$$\frac{A_{k-1}}{D_{k-1}} - \frac{A_{k+1}}{D_{k+1}}$$

by means of a very simple artifice. Replace a_{kk} by $a_{kk}+t$, and let the new values of the various determinants be denoted by accents. Thus, say

$$\xi'_{k+1} = \xi_{k+1} + t\eta_k, \quad D'_k = tD_{k-1}, \quad D'_{k+1} = D_{k+1} + tE_k.$$

Further, since $D_k = 0$,

we have $$-\xi_k^2 = D_{k-1}A_k, \quad D_{k+1}A_k - \xi^2_{k+1} = 0,$$

so that $$\xi_k^2 = -\xi^2_{k+1}D_{k-1}/D_{k+1}.$$

Thus
$$\frac{A_{k-1}}{D_{k-1}} - \frac{A'_{k+1}}{D'_{k+1}} = \frac{\xi_k^2}{D_{k-1}D'_k} + \frac{\xi'^2_{k+1}}{D'_k D'_{k+1}}$$
$$= \frac{1}{tD_{k-1}}\left[\frac{\xi'^2_{k+1}}{D'_{k+1}} - \frac{\xi^2_{k+1}}{D_{k+1}}\right].$$

If now we take the limit of this equation, when t approaches zero, we obtain

$$\frac{A_{k-1}}{D_{k-1}} - \frac{A_{k+1}}{D_{k+1}} = \frac{1}{D_{k-1}}\left[\frac{2\xi_{k+1}\eta_k}{D_{k+1}} - \frac{\xi^2_{k+1}E_k}{D^2_{k+1}}\right]$$
$$= \frac{2\xi_{k+1}}{D_{k-1}D_{k+1}}\left(\eta_k - \frac{E_k\xi_{k+1}}{2D_{k+1}}\right) \quad \ldots\ldots\ldots\ldots(2),$$

which expresses this difference as the product of two factors, instead of the sum of two squares: the product may of course be expressed in terms of two squares, in the way indicated at the beginning of this article.

Naturally if there are two or more consecutive zero determinants in the sequence $D_1, D_2, \ldots, D_n$, a similar limiting process could be used to express the corresponding terms; but it seems hardly worth while to write out a general formula, which is necessarily rather complicated and is not needed in the future work.

* See p. 354, formula 17 of the paper quoted in Art. 38; a reproduction is given in Scott's *Determinants*, pp. 150, 151. The reader may also consult Frobenius, *Journal für Math.* Bd. 114, 1895, p. 187.

9. Positive quadratic forms and Sylvester's law of inertia.

We have already met with an example (see small type, Art. 6) of a quadratic form which is positive for *all real non-zero* values of the variables. Such forms are called *positive definite forms*: or sometimes simply *positive forms*. By changing the sign of every coefficient we obtain a *negative definite form*.

Example: $x^2 - xy + y^2$ is a *positive* form; $-x^2 + xy - y^2$ is a *negative* form.

It is often important* to be able to recognise whether a given quadratic form possesses the property of being definite; this can be settled by the aid of the formulae of Art. 8.

Supposing that $D_0, D_1, D_2, \ldots, D_r$ are all different from zero, it will follow that

$$A = \frac{A_r}{D_r} + \sum_{k=1}^{r} \frac{\xi_k^2}{D_{k-1}D_k} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

by using the equations leading up to (1) in Art. 8. Now, since D_r is not zero, the n variables

$$\xi_1, \xi_2, \ldots, \xi_r, x_{r+1}, x_{r+2}, \ldots, x_n,$$

are *linearly independent*; and therefore A_r must be definite in the variables $x_{r+1}, x_{r+2}, \ldots, x_n$. Hence the coefficient of x^2_{r+1} in A_r cannot vanish; for if it did A_r would be zero for the values

$$x_{r+1} = 1, \quad x_{r+2} = 0, \quad \ldots, \quad x_n = 0.$$

Now this coefficient is D_{r+1}; and so we have deduced from the hypothesis that $D_0, D_1, D_2, \ldots, D_r$ do not vanish, the conclusion that D_{r+1} is also different from zero. But $D_0 = 1$, $D_1 = a_{11}$, which cannot vanish if A is to be definite; and therefore in succession we deduce that $D_2, D_3, \ldots, D_n$ cannot be zero†.

Thus equation (1) Art. 8 can be applied if A is definite; and so the n products $D_0D_1, D_1D_2, D_2D_3, \ldots, D_{n-1}D_n$ must have the same sign as A. Hence, since $D_0 = 1$, we have the conditions:—

(i) If A is positive, $D_1, D_2, D_3, \ldots, D_n$ are all positive.

(ii) If A is negative, $D_1, D_3, D_5, \ldots$ are negative, $D_2, D_4, D_6, \ldots$ are positive.

These conditions have been proved to be *necessary* and they are obviously *sufficient* also.

* For example in the theory of maxima and minima of a function of several variables; in discussing the stability of a state of equilibrium; and in similar problems.

† Compare footnote p. 11.

A form is said to be semidefinite, if it cannot change sign but may be zero for real non-zero values of the variables. Thus in equation (1) above it may happen that $D_{r+1}=0$; but if so, A_r *must be independent of* x_{r+1}. For otherwise method (ii) of Art. 7 would express A_r in the form

$$2y_r y_{r+1} + f(x_{r+2}, x_{r+3}, \ldots, x_n),$$

which can be made to have either sign, contrary to hypothesis.

Thus if $D_{r+1}=0$, all the following determinants $D_{r+2}, D_{r+3}, \ldots, D_n$ must also vanish; and the preceding determinants must satisfy conditions (i) or (ii) above. But these results are not sufficient to ensure that the sign of A_r/D_r shall be always the same as the sign attributed to A. The simplest method is to alter the order of the x's, taking next to x_r a variable which does actually appear in A_r: proceeding thus we obtain a set of determinants which must satisfy conditions (i) or (ii) above.

Since a semidefinite form is definite in a smaller number of variables, we may also express it first in terms of ρ variables (where ρ is its rank), and then conditions (i) or (ii) must apply.

Example 1. $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$

is positive if a, C, Δ are positive (compare Ex. 1, Art. 8).

Example 2. $x^2+y^2+cz^2+2fyz+2fzx+2xy$,

gives the series of determinants

$$D_1=1, \quad D_2=0, \quad D_3=0.$$

Thus the form may be semidefinite; to test this, take the variables in the order x, z, y. Then the determinants are

$$D_1=1, \quad D_2=c-f^2, \quad D_3=0;$$

and so the form is semidefinite if $c-f^2$ is positive; its rank is 2.

In a maximum or minimum problem, if the quadratic terms are found to be semidefinite, it is often difficult to decide whether a true extreme value occurs; and the rule commonly adopted (*e.g.* Edwards, *Differential Calculus*, 3rd edition, § 497) may lead to error.

For example consider $(x+y-y^2)(x+y-2y^2)$, in the neighbourhood of $x=0$, $y=0$; the quadratic terms are $(x+y)^2$, and are semidefinite; when $x+y=0$, the expression reduces to $2y^4$, which has the same sign as $(x+y)^2$. Thus the rule would lead us to infer a minimum; but nevertheless the expression may be made negative, by taking $x+y=\frac{3}{2}y^2$, so that $x=0$, $y=0$ does *not* give a true minimum of the expression. A sufficient method for most cases in two variables is the following. Suppose $z=f(x, y)$ the given

function, and that the values $x=a$, $y=b$, $z=c$ satisfy the preliminary tests for a maximum or minimum, the quadratic terms proving to be a perfect square; then examine the form of the curve $f(x, y)=c$, near the point $x=a$, $y=b$. If the curve has *real* branches through the point, $z=c$ does *not* give a true maximum or minimum; but if the point is isolated, $z=c$ does give a true maximum or minimum. Of course Newton's diagram* is usually the best guide in complicated cases.

Sylvester's law of inertia states that in whatever way a quadratic form is reduced to a sum of squares, the total number of positive squares and the total number of negative squares are always the same, provided that the squares are *real* and *independent.*

Examples:—x^2+xy+y^2 always gives *two* positive squares (see p. 16, small type); $x^2-3xy+2y^2$ always gives one positive and one negative square.

But the fact that we may write

$$3(x^2+y^2)=x^2+y^2+(x+y)^2+(x-y)^2$$

does not contradict the theorem, because $x+y$, $x-y$, x, y are not four independent variables.

Neither is the equation

$$2(x^2-y^2)=(x+iy)^2+(x-iy)^2$$

contrary to the theorem, because complex coefficients have been introduced on the right-hand side.

The proof of the theorem is very simple; suppose that the quadratic form is of rank n and is expressed in terms of n *independent* variables $x_1, x_2, \ldots, x_n$. Let it then be reduced to the two forms

$$A_1X_1^2+A_2X_2^2+\ldots+A_kX_k^2-(A_{k+1}X^2_{k+1}+\ldots+A_nX_n^2),$$
$$B_1Y_1^2+B_2Y_2^2+\ldots+B_lY_l^2-(B_{l+1}Y^2_{l+1}+\ldots+B_nY_n^2),$$

where $X_1, X_2, \ldots, X_n$ are real linearly independent linear functions of $x_1, x_2, \ldots, x_n$ and each of the coefficients A is positive; corresponding statements being true of the Y's and B's.

Since the two expressions for the quadratic form are identically equal, we have

$$(A_1X_1^2+\ldots+A_kX_k^2)+(B_{l+1}Y^2_{l+1}+\ldots+B_nY_n^2)$$
$$=(B_1Y_1^2+\ldots+B_lY_l^2)+(A_{k+1}X^2_{k+1}+\ldots+A_nX_n^2).$$

Now, if $k<l$, the $n+k-l$ equations

$$X_1=0, \quad X_2=0, \quad \ldots, \quad X_k=0,$$
$$Y_{l+1}=0, \quad Y_{l+2}=0, \quad \ldots, \quad Y_n=0,$$

* Chrystal's *Algebra*, Ch. xxx, § 22; for applications to the present problem the reader may consult Scheeffer, *Math. Annalen*, Bd. 35, pp. 572–6.

being fewer than n in number, can be satisfied by real non-zero values of $x_1, x_2, \ldots, x_n$; and for these values

$$(B_1 Y_1^2 + \ldots + B_l Y_l^2) + (A_{k+1} X^2_{k+1} + \ldots + A_n X_n^2) = 0.$$

Since all the coefficients in the last equation are positive, and since the values of the variables are *real*, this equation can only be satisfied if

$$Y_1 = 0, \quad Y_2 = 0, \quad \ldots, \quad Y_l = 0,$$
$$X_{k+1} = 0, \quad X_{k+2} = 0, \quad \ldots, \quad X_n = 0;$$

that is, we have found real non-zero values of $x_1, x_2, \ldots, x_n$ which satisfy

$$X_1 = 0, \quad X_2 = 0, \quad \ldots, \quad X_n = 0.$$

Now this is impossible, since the functions $X_1, X_2, \ldots, X_n$ are supposed linearly independent; consequently k cannot be less than l.

In like matter l cannot be less than k; so that we must have

$$k = l,$$

which is Sylvester's law of inertia. (*Phil. Mag.* series 4, vol. IV, 1852, p. 142: *Coll. Papers*, vol. I, pp. 380, 381.)

Thus, under the group of *real* linear substitutions, a quadratic form has two invariants, which may be conveniently taken to be (i) the *rank* (Art. 5) and (ii) the *signature*, which is the difference $k-(n-k) = 2k-n$, in the above case. For a positive form, the signature is equal to the rank.

10. Effect of a linear substitution on a quadratic form, and on its determinant.

For the sake of brevity, we shall take the case of three variables only, although the methods and results apply to any number of variables. (Sylvester, *Phil. Mag.* series 4, vol. I, 1851, p. 295: *Coll. Papers*, vol. I, p. 241.)

Suppose the form to be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

and let the substitution be

$$x = l_1\xi + m_1\eta + n_1\zeta,$$
$$y = l_2\xi + m_2\eta + n_2\zeta,$$
$$z = l_3\xi + m_3\eta + n_3\zeta.$$

The form then becomes

$$a'\xi^2 + b'\eta^2 + c'\zeta^2 + 2f'\eta\zeta + 2g'\zeta\xi + 2h'\xi\eta$$

where $$a' = al_1^2 + bl_2^2 + cl_3^2 + 2fl_2l_3 + 2gl_3l_1 + 2hl_1l_2,$$

$$f' = am_1n_1 + bm_2n_2 + cm_3n_3 + f(m_2n_3 + m_3n_2) + g(m_3n_1 + m_1n_3) + h(m_1n_2 + m_2n_1),$$

with similar values for the other coefficients. Write for brevity

$$L_1 = al_1 + hl_2 + gl_3,$$
$$L_2 = hl_1 + bl_2 + fl_3,$$
$$L_3 = gl_1 + fl_2 + cl_3,$$

with similar meanings for M_1, M_2, M_3 and N_1, N_2, N_3. Then we have

$$a' = l_1L_1 + l_2L_2 + l_3L_3,$$
$$h' = l_1M_1 + l_2M_2 + l_3M_3 = m_1L_1 + m_2L_2 + m_3L_3;$$

with similar equations for the other coefficients.

Thus the determinant of the new form is

$$\begin{vmatrix} a', & h', & g' \\ h', & b', & f' \\ g', & f', & c' \end{vmatrix} = \begin{vmatrix} l_1, & l_2, & l_3 \\ m_1, & m_2, & m_3 \\ n_1, & n_2, & n_3 \end{vmatrix} \times \begin{vmatrix} L_1, & L_2, & L_3 \\ M_1, & M_2, & M_3 \\ N_1, & N_2, & N_3 \end{vmatrix}.$$

If now we introduce the values of L_1, L_2, L_3, etc. in terms of l_1, l_2, l_3, etc., we get

$$\begin{vmatrix} L_1, & L_2, & L_3 \\ M_1, & M_2, & M_3 \\ N_1, & N_2, & N_3 \end{vmatrix} = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \times \begin{vmatrix} l_1, & l_2, & l_3 \\ m_1, & m_2, & m_3 \\ n_1, & n_2, & n_3 \end{vmatrix}.$$

Hence we find

$$\begin{vmatrix} a', & h', & g' \\ h', & b', & f' \\ g', & f', & c' \end{vmatrix} = K^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots(1),$$

where K is the determinant

$$\begin{vmatrix} l_1, & l_2, & l_3 \\ m_1, & m_2, & m_3 \\ n_1, & n_2, & n_3 \end{vmatrix}.$$

Now, although this investigation relates to three variables only, it will be seen without difficulty that the method and the result are both valid for any number of variables.

In general, if a form A, expressed in terms of $x_1, x_2, x_3, \ldots, x_n$, becomes A', in terms of $y_1, y_2, y_3, \ldots, y_n$, we shall have

$$|A'| = K^2 |A| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where K is the determinant represented in the Jacobian notation by

$$\frac{\partial (x_1, x_2, x_3, \ldots, x_n)}{\partial (y_1, y_2, y_3, \ldots, y_n)}.$$

Consider now the value of the determinant

$$\begin{vmatrix} a', & h', & g', & l_1 \\ h', & b', & f', & m_1 \\ g', & f', & c', & n_1 \\ l_1, & m_1, & n_1, & 0 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

obtained by "bordering" the transformed determinant with a row of the coefficients in the substitution.

This is the determinant of the quadratic form (in the variables ξ, η, ζ, t)

$$a'\xi^2 + b'\eta^2 + c'\zeta^2 + 2f'\eta\zeta + 2g'\zeta\xi + 2h'\xi\eta + 2t\,(l_1\xi + m_1\eta + n_1\zeta)$$
$$= ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2tx.$$

Hence, by equation (2) above,

$$\begin{vmatrix} a', & h', & g', & l_1 \\ h', & b', & f', & m_1 \\ g', & f', & c', & n_1 \\ l_1, & m_1, & n_1, & 0 \end{vmatrix} = K_1^2 \begin{vmatrix} a, & h, & g, & 1 \\ h, & b, & f, & 0 \\ g, & f, & c, & 0 \\ 1, & 0, & 0, & 0 \end{vmatrix} \quad \ldots\ldots\ldots(4),$$

where $$K_1 = \frac{\partial (x, y, z, t)}{\partial (\xi, \eta, \zeta, t)} = \frac{\partial (x, y, z)}{\partial (\xi, \eta, \zeta)} = K.$$

Thus we find that the determinant (3) is equal to

$$-K^2 \begin{vmatrix} b, & f \\ f, & c \end{vmatrix}$$

and so is a multiple of one of the principal first minors of the original determinant.

If, instead of bordering with l_1, m_1, n_1, we border with $l_1 + \theta l_2$, $m_1 + \theta m_2$, $n_1 + \theta n_2$, we see that the determinant so obtained is equal to

$$K^2 \begin{vmatrix} a, & h, & g, & 1 \\ h, & b, & f, & \theta \\ g, & f, & c, & 0 \\ 1, & \theta, & 0, & 0 \end{vmatrix}$$

$$= K^2 \left[-\begin{vmatrix} b, & f \\ f, & c \end{vmatrix} + 2\theta \begin{vmatrix} h, & g \\ f, & c \end{vmatrix} - \theta^2 \begin{vmatrix} a, & g \\ g, & c \end{vmatrix} \right].$$

Now equate coefficients of 2θ and we find

$$\begin{vmatrix} a', & h', & g', & l_1 \\ h', & b', & f', & m_1 \\ g', & f', & c', & n_1 \\ l_2, & m_2, & n_2, & 0 \end{vmatrix} = K^2 \begin{vmatrix} h, & g \\ f, & c \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots(5).$$

The determinant on the left of equation (5) can be written out at length in the form

$$-l_1l_2 \begin{vmatrix} b', & f' \\ f', & c' \end{vmatrix} - m_1m_2 \begin{vmatrix} a', & g' \\ g', & c' \end{vmatrix} - n_1n_2 \begin{vmatrix} a', & h' \\ h', & b' \end{vmatrix}$$
$$+ (m_1n_2 + m_2n_1) \begin{vmatrix} a', & g' \\ h', & f' \end{vmatrix} - (n_1l_2 + n_2l_1) \begin{vmatrix} h', & g' \\ b', & f' \end{vmatrix}$$
$$+ (l_1m_2 + l_2m_1) \begin{vmatrix} h', & g' \\ f', & c' \end{vmatrix}$$

which is a *linear* function of the first minors in the transformed determinant. Similarly we can expand the determinant (3).

Thus, since K is not zero*, we can state the theorem:—*The original determinant is a multiple of the transformed determinant; and every first minor of the original determinant is a linear function of the first minors of the transformed determinant. And the converse theorem is also true,* for we can solve for ξ, η, ζ and write

$$\xi = \lambda_1 x + \lambda_2 y + \lambda_3 z,$$
$$\eta = \mu_1 x + \mu_2 y + \mu_3 z,$$
$$\zeta = \nu_1 x + \nu_2 y + \eta_3 z;$$

then the argument can be reversed, writing λ_1, λ_2, λ_3 instead of l_1, m_1, n_1, etc.

The same result is true for minors of the second order, or of any higher order (when there are more than three variables). To deal with a second minor, we should have to use a determinant bordered with two rows and columns, such as

$$\begin{vmatrix} a', & h', & g', & l_1, & l_2 \\ h', & b', & f', & m_1, & m_2 \\ g', & f', & c', & n_1, & n_2 \\ l_1, & m_1, & n_1, & 0, & 0 \\ l_2, & m_2, & n_2, & 0, & 0 \end{vmatrix};$$

and so on for minors of any order.

* If K were zero, ξ, η, ζ would not be linearly independent functions of x, y, z.

CHAPTER III.

REDUCTION AND CLASSIFICATION OF A FAMILY OF QUADRATIC FORMS.

11. Invariant-factors. Definitions and abbreviations.

If we have a pair of quadratic forms A, B which contain the same n variables, we can construct from A and B the *family of forms** $(\lambda A - B)$; and if the determinant of the family, denoted by $|\lambda A - B|$, is split up into factors (linear in λ), then it is well-known that each factor is an invariant for all linear substitutions on the variables contained in A, B†. A proof of this theorem will be given in the following article. Further, if the factors of the determinant are all distinct, they will form a complete set of invariants; so that if A' and B' are another pair of forms such that $|\lambda A' - B'|$ has the same factors as $|\lambda A - B|$, then A, B can be transformed into A', B' respectively by the same linear substitution. But, on the other hand, if some of the factors are repeated, it does not follow, as a rule, that the set of invariants is complete without further information. As an example, consider (i) a family of conics having single contact, (ii) a family having double contact; in each case the determinant is of the form $(\lambda - a)^2 (\lambda - b)$. Thus the mere knowledge that the determinant has a squared factor does not distinguish between single and double contact. In fact, this distinction is not at all easy by the ordinary methods of invariants (Salmon, *Conic Sections*, Art. 378 *a*).

Suppose that a factor $(\lambda - c)$ appears raised to the power l_0 in the determinant; we must now examine the first minors and let us suppose that l_1 is the index of that power of $(\lambda - c)$ which will divide them *all*; the value of l_1 may be found from the H.C.F. of the first minors. In the same way let $l_2, l_3, \ldots$ be the indices of $(\lambda - c)$ found from all the second, third, ..., minors; and let l_r be the first index in the

* The word *family* is here adopted as the equivalent of the German *Schaar*.

† Cf. Salmon, *Conic Sections*, Chap. XVIII, Art. 371, and *Geometry of Three Dimensions*, Chap. IX, Art. 200.

sequence which is zero, so that the rth minors are not all divisible by $(\lambda - c)$, although all preceding minors are so divisible. It will be proved later that the aggregate of all factors such as $(\lambda - c)$ together with the corresponding indices $l_0, l_1, l_2, \ldots, l_{r-1}$ will give a complete set of invariants for the family of quadratic forms. But, for several reasons, it is preferable to use, instead of the indices $l_0, l_1, l_2, \ldots, l_{r-1}$, their successive differences $e_1 = l_0 - l_1$, $e_2 = l_1 - l_2$, $\ldots$, $e_r = l_{r-1}$. Then the powers $(\lambda - c)^{e_1}, (\lambda - c)^{e_2}, \ldots, (\lambda - c)^{e_r}$ of $(\lambda - c)$ are called *invariant-factors* (*Elementartheiler** or, in modern spelling, *Elementarteiler*) to the base $(\lambda - c)$ of the determinant of the family of forms; and it will be proved that the family can be reduced to a type which depends on the invariant-factors only, so that if two families have the same invariant-factors, we can transform the one into the other by means of a linear substitution. This is, indeed, what we mean by saying that the invariant-factors are a complete set of invariants for the family.

It should be noticed that the sum of the indices of the invariant-factors to base $(\lambda - c)$ is l_0; and consequently the sum of all indices of the invariant-factors of the family is n, because the degree of the determinant is equal to the number of variables. Further, the first derivate of the determinant $|\lambda A - B|$ can be expressed as a sum of first minors, so that $(l_0 - 1)$ is not less than l_1: that is, e_1 is not less than 1. Similarly each of the indices $e_2, e_3, \ldots, e_r$ is not less than 1.

To make the definitions clearer we shall illustrate them by some examples.

(i) $A = x^2 + 2yz$, $B = 2xy$, $|\lambda A - B| = \begin{vmatrix} \lambda, & -1, & 0 \\ -1, & 0, & \lambda \\ 0, & \lambda, & 0 \end{vmatrix}$.

The determinant is equal to $-\lambda^3$, but one first minor (corresponding to the last row and column) is -1; thus the indices l are given by $l_0 = 3$, $l_1 = 0$ and so $e_1 = 3$. There is thus only one invariant-factor, namely λ^3.

(ii) $A = x^2 + 2yz$, $B = y^2$, $|\lambda A - B| = \begin{vmatrix} \lambda, & 0, & 0 \\ 0, & -1, & \lambda \\ 0, & \lambda, & 0 \end{vmatrix}$.

* Introduced in this form and with this name by Weierstrass in his second paper quoted in Art. 38; but 17 years earlier Sylvester (*Phil. Mag.* series 4, vol. I, 1851, pp. 119—140; *Coll. Papers*, vol. I, pp. 219—240) had used the powers $(\lambda - c)^l$ to classify conics and quadrics. However Weierstrass was the first to shew that any non-singular family can be reduced to a canonical type, each invariant-factor giving a group of terms in the reduced forms. Sylvester gave canonical types for $n = 3, 4$ and afterwards found the number of types for all values of n up to 11 (see Art. 23); but he had no general method of reduction nor for finding the canonical types (which of course are not so directly connected with the indices l_k as with the indices e_k).

The determinant is again $-\lambda^3$, but here every first minor is divisible by λ (in fact, all but two are zero, and these two are equal to $-\lambda$ and $-\lambda^2$); and one second minor is -1; so that $l_0=3$, $l_1=1$, $l_2=0$, giving $e_1=2$, $e_2=1$. Thus there are two invariant-factors λ^2, λ.

(iii) $A=x^2+y^2+z^2$, $B=z^2$, $|\lambda A-B| = \begin{vmatrix} \lambda, & 0, & 0 \\ 0, & \lambda, & 0 \\ 0, & 0, & \lambda-1 \end{vmatrix}$.

The determinant is $\lambda^2(\lambda-1)$; all the first minors are divisible by λ, but not by $(\lambda-1)$; and the second minors are not all divisible by λ. Thus, for the base λ, we get $l_0=2$, $l_1=1$, $l_2=0$, and $e_1=1$, $e_2=1$; for the base $(\lambda-1)$, we have $l_0=1$, $l_1=0$, so that $e_1=1$. Hence the invariant-factors are λ, λ, $(\lambda-1)$.

(iv) $A=x^2+y^2+z^2$, $B=x^2+2y^2+3z^2$,

$$|\lambda A-B| = \begin{vmatrix} \lambda-1, & 0, & 0 \\ 0, & \lambda-2, & 0 \\ 0, & 0, & \lambda-3 \end{vmatrix}.$$

Since the factors of the determinant are all different, they are the invariant-factors: namely $(\lambda-1)$, $(\lambda-2)$, $(\lambda-3)$.

We have thus a complete theory for classifying families of quadratic forms; but it may be urged that the invariant-factors are not rational invariants, because they involve the solution of an algebraic equation. To meet this objection we may use a slightly different set of invariants (introduced by Kronecker). Thus, suppose that D_0 is the determinant of the family; that D_1 is the H. C. F. of all the first minors of D_0; that D_2 is the H. C. F. of all the second minors; and so on. Then Kronecker's invariants are

$$E_1=D_0/D_1,\ E_2=D_1/D_2,\ E_3=D_2/D_3,\ \ldots,\ E_r=D_{r-1}/D_r.$$

It is easy to see that if the quotients $E_1, E_2, E_3, \ldots, E_r$ are the same for two families of forms, then the invariant-factors are also the same; for the invariant-factors can be obtained by dividing

$$E_1,\ E_2,\ E_3,\ \ldots,\ E_r$$

into their factors. Thus we have a *rational* process for determining the identity (save for a linear substitution) of two given families; and methods of finding the substitution rationally have been given by Frobenius, Kronecker and Landsberg. But we shall not concern ourselves with this method, and shall suppose the invariant-factors known in all cases.

To make the distinction clearer between Kronecker's invariants and the invariant-factors, take the case

(v) $A=2(xz-yt)$, $B=2(yz+xt)+x^2-y^2$.

Here
$$|\lambda A - B| = \begin{vmatrix} -1, & 0, & \lambda, & -1 \\ 0, & 1, & -1, & -\lambda \\ \lambda, & -1, & 0, & 0 \\ -1, & -\lambda, & 0, & 0 \end{vmatrix}.$$

The determinant is $(\lambda^2+1)^2$ and the H.C.F. of the first minors is 1. So that Kronecker's invariant E_1 is $(\lambda^2+1)^2$; but the separate invariant-factors would be $(\lambda+i)^2$ and $(\lambda-i)^2$.

We now enunciate a theorem of fundamental importance, reserving part of the proof until Art. 24:—

If a family of forms can be separated into two parts, which have no variables in common, the invariant-factors of the family are the same as those of the parts, taken together.

In fact, if $\lambda A - B = (\lambda A_1 - B_1) + (\lambda A_2 - B_2)$, where the families 1, 2 have no variables in common, it is clear that the determinant $|\lambda A - B|$ can be written in the form

$$\left|\begin{array}{c|c} \Delta_1 & \omega \\ \hline \omega & \Delta_2 \end{array}\right|$$

where $\Delta_1 = |\lambda A_1 - B_1|$, $\Delta_2 = |\lambda A_2 - B_2|$, and the ω's represent rectangular blocks of zeros.

It is at once obvious that any minor corresponding to elements in either block ω is zero; and consequently any non-zero minor in $|\lambda A - B|$ is the product of a minor belonging to Δ_1 by a minor belonging to Δ_2. From this it is evident that if $(\lambda - c)$ is a factor of Δ_1, but not a factor of Δ_2, then the set of invariant-factors of Δ_1 to base $(\lambda - c)$ is also a set of invariant-factors of $|\lambda A - B|$; and similarly for invariant-factors of Δ_2 to a base which is not a factor of Δ_1. However, it is not so obvious that, if both Δ_1 and Δ_2 have invariant-factors to base $(\lambda - c)$, then $|\lambda A - B|$ has the *same* invariant-factors*; but the theorem is nevertheless true, as the reader may easily see by considering a few special cases; a formal proof will be found in Art. 24.

In the lists of the reduced forms we use a symbol called the *characteristic* by Segre and other writers, which will be best explained by an example; supposing that we had a set of invariant-factors

$$(\lambda - a)^2, \quad (\lambda - a), \quad (\lambda - b)^3, \quad (\lambda - c),$$

they would be denoted by [(21) 31]. Thus the indices in round brackets ( ) belong to the same factor, and the square brackets [ ] surround all the indices.

* **For instance if Δ_1, Δ_2 had each an invariant-factor $(\lambda - c)^2$, it is not evident that $|\lambda A - B|$ might not have the two $(\lambda - c)^3$, $(\lambda - c)$, instead of $(\lambda - c)^2$, $(\lambda - c)^2$.**

With the aid of this symbol, we may classify the three cases of Art. 3 as follows:—

(i) [11]; (ii) [2]; (iii) [(11)].

And the five examples above have the characteristics

(i) [3]; (ii) [(21)]; (iii) [(11) 1]; (iv) [111]; (v) [22].

12. Proof that invariant-factors are actually invariants under any linear substitution.

Apply the results of Art. 10 to the determinant $|\lambda A - B|$; then *since the linear substitution is independent of* λ, it is clear that (if A, B become A', B' after the substitution is applied) the determinants $|\lambda A - B|$ and $|\lambda A' - B'|$ can differ only by a factor which is independent of λ. Thus if $|\lambda A - B|$ contains a factor $(\lambda - c)$ raised to the power of l_0, $|\lambda A' - B'|$ contains the same factor raised to the same power; and conversely.

Again, by the same article, any first minor of $|\lambda A' - B'|$ is expressible as a linear combination of the first minors of $|\lambda A - B|$; and the coefficients in the expression depend only on the first minors of the linear substitution, so that they are independent of λ. Thus if $(\lambda - c)^{l_1}$ is a factor of all the first minors of $|\lambda A - B|$, it will also be a factor of all first minors of $|\lambda A - B'|$; and conversely.

The same argument can be applied to the minors of any order; and we deduce that the factor $(\lambda - c)$ and the sequence of indices

$$l_0, \quad l_1, \quad l_2, \quad \ldots, \quad l_{r-1}$$

are the same for the determinants $|\lambda A - B|$ and $|\lambda A' - B'|$. In other words, the factor and its indices are invariants under any linear substitution applied to the family of forms. But the differences

$$e_1 = l_0 - l_1, \quad e_2 = l_1 - l_2, \quad \ldots, \quad e_r = l_{r-1}$$

are equally invariants, since they can be deduced from $l_0, l_1, \ldots, l_{r-1}$ (and conversely). Thus *the invariant-factors are also invariants under any linear substitution applied to the family of forms.*

Although the foregoing discussion refers only to the separate invariant-factors, yet it is easily seen that the same proof establishes the fact that *Kronecker's quotients are also invariants.* Indeed E_1 is simply the product of all such invariant-factors as $(\lambda - c)^{e_1}$; and so on.

13. Reduction of two quadratic forms in three variables.

The method given here is only a special case of the general process for n variables described in Art. 20: but a beginner will probably find

it easier to become familiar with the special case first. We shall accompany the algebraic reduction with a geometrical explanation, taking the three variables as homogeneous coordinates of points in a plane; so that the quadratic forms represent conics, when equated to zero.

If A and B are the given quadratic forms suppose that $(\lambda - c)$ has been found to be a factor of the determinant $|\lambda A - B|$. Then $(cA - B)$ contains at most *two* independent variables* (by Art. 5); and let the variables x_1, x_2, x_3 be first chosen so that x_1 does not occur in $(cA - B)$. Then the problem divides itself into three, according as the form $(cA - B)$ contains two variables, one variable, or is identically zero.

§ 1. First suppose that $(cA - B)$ contains two variables, so that

$$cA - B = f(x_2, x_3), \quad A = g(x_1, x_2, x_3) \ldots\ldots\ldots\ldots (1).$$

Geometrically speaking we have found c so as to make $cA - B = 0$ represent a pair of straight lines, intersecting in **P**, say; and then **P** has been taken as a vertex of the fundamental triangle.

The next step is to see *if x_1^2 is present in A*; if so, the method (i) of Art. 7 can be used to isolate all the terms in x_1, by writing†

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \quad A = (y_1^2/a_{11}) + h(x_2, x_3).$$

Thus we have found a set of variables such that

$$A = y_1^2/a_{11} + h(x_2, x_3), \quad cA - B = f(x_2, x_3) \ldots\ldots\ldots (2).$$

The reduction can now be completed by applying the methods of Arts. 2, 3 to $f(x_2, x_3)$ and $h(x_2, x_3)$; and there are three possible cases, as found in those articles.

Geometrically, the presence of x_1^2 in A means that *the point* **P** *is not on the conic* $A = 0$; so that the polar of **P** (which is the same line for all the conics $\lambda A - B = 0$) will not pass through **P**, and so may be taken as the opposite side of the triangle of reference; and the remainder of the discussion depends on the involution intercepted by the conics on this side. The three cases will be sufficiently clear from Figs. 1—3 on p. 32. In the first and second cases, a second pair of lines $c'A - B = 0$ can be found, intersecting in **Q**, which is a

* If the determinant $|A|$ is zero, $|\lambda A - B|$ may have no factor such as $(\lambda - c)$; if so, use A, B instead of $(cA - B)$ and A respectively. It frequently happens that $|\lambda A - B|$ has at least two distinct factors $(\lambda - c)$, $(\lambda - c')$; and then the work is shortened by taking $(cA - B)$ and $(c'A - B)$ as the forms to start with (compare Ex. 1, Art. 16).

† We take $A = a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \ldots$.

second vertex of the triangle of reference. In the first case the third vertex **O** is the pole of the line **PQ**; in the second it is the point of contact of the other tangent from **P** to the conic $A = 0$. In the third case, **Q** can be taken anywhere on the polar of **P**, and **O** is then the harmonic conjugate of **Q** with respect to the conics.

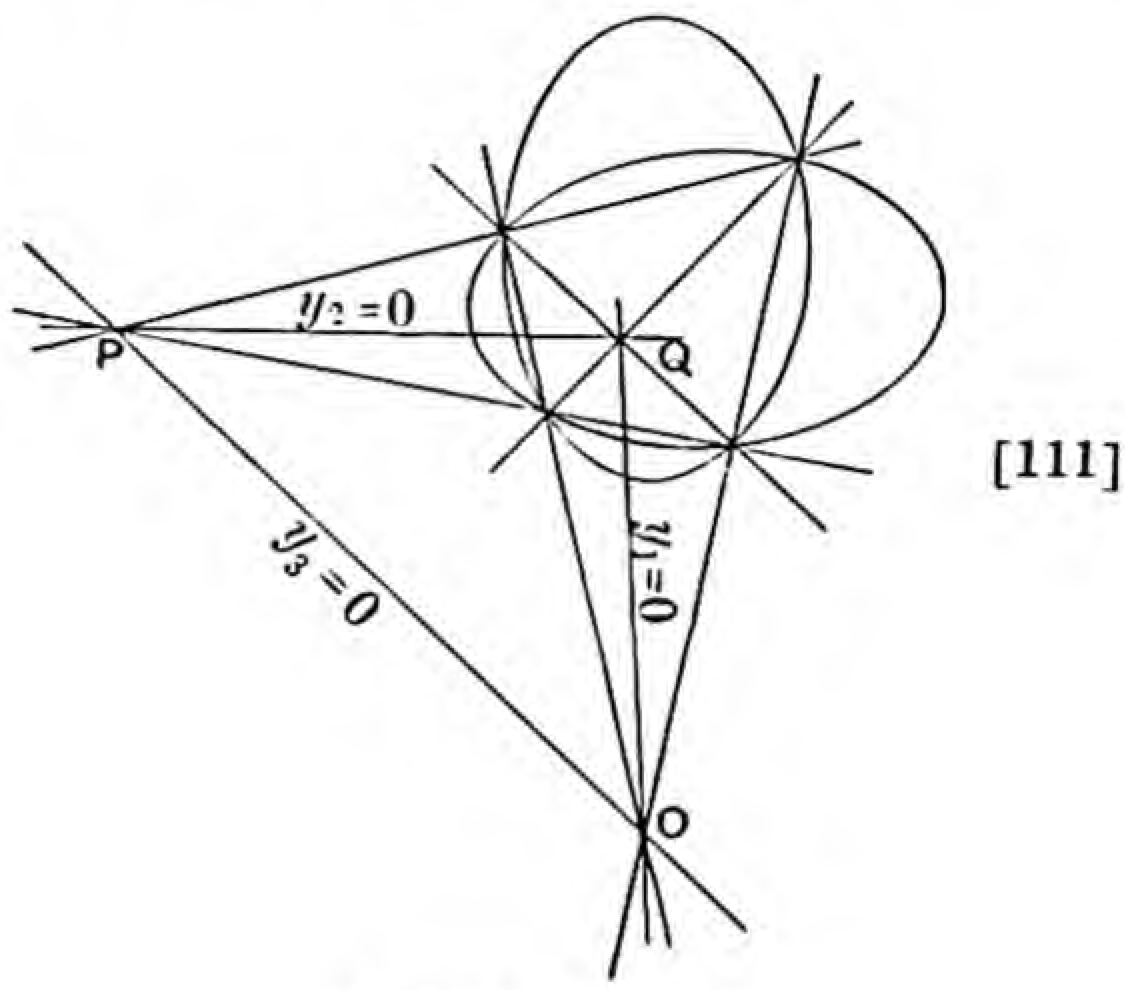


Fig. 1.

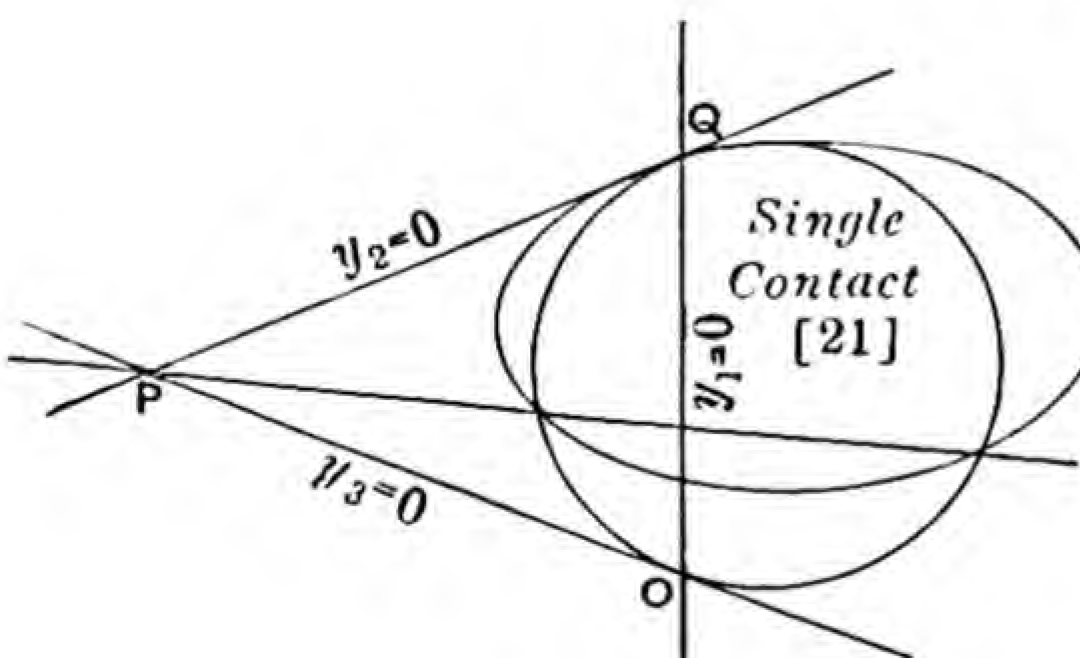


Fig. 2.

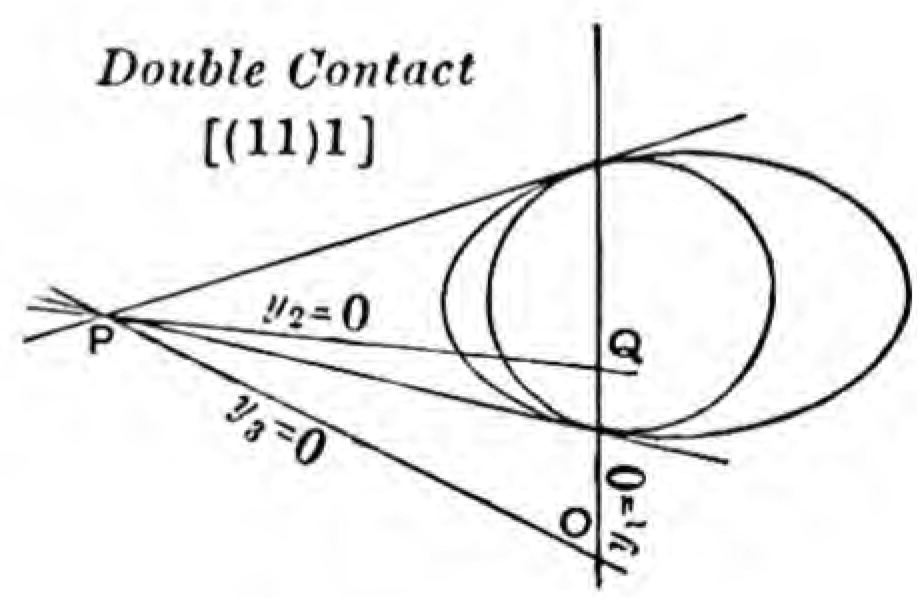


Fig. 3.

But, *if* x_1^2 *is not present in* A, (so that $a_{11} = 0$), we may suppose a_{12} different from zero*, and then method (ii) of Art. 7 is to be applied to A. We obtain then

$$A = 2y_1y_2 + ax_3^2,$$

where
$$y_1 = (a_{21}x_1 + a_{22}x_2 + a_{23}x_3) - \tfrac{1}{2}\frac{a_{22}}{a_{12}}(a_{12}x_2 + a_{13}x_3),$$

$$y_2 = \frac{1}{a_{12}}(a_{12}x_2 + a_{13}x_3), \quad a = -\frac{|A|}{a_{12}^2}.$$

* For, provided that x_1 is actually present in A, one or other of a_{12}, a_{13} must not vanish; and we suppose the suffixes chosen so that a_{12} is not zero. Of course if x_1 is not present in A, the problem depends only on the reduction of forms in *two* variables x_2, x_3.

Substitute now $x_2 = y_2 - (a_{13}/a_{12})\,x_3$ in $(cA - B)$ and we obtain the forms

$$A = 2y_1y_2 + ax_3^2, \quad cA - B = f_1(y_2, x_3)\ldots\ldots\ldots\ldots\ldots(3).$$

The geometrical meaning of this transformation is that **P** is now on the conics $\lambda A - B = 0$, and $y_2 = 0$ is the polar of **P** (that is to say, the common tangent to the conics at **P**); while $y_1 = 0$ is the tangent at the other intersection of the line $x_3 = 0$ with the conic $A = 0$.

We must next examine (3), to see *if x_3^2 is present in* $(cA - B)$; if so, we may apply method (iii) of Art. 7, and obtain*

$$cA - B = b_2y_2^2 + b_3(x_3 + ky_2)^2 = b_2y_2^2 + b_3y_3^2.$$

Substitute $x_3 = y_3 - ky_2$ in A and we get

$$A = 2y_1y_2 + a(y_3 - ky_2)^2 = 2y_1'y_2 + ay_3^2,$$

where $$y_1' = y_1 + \tfrac{1}{2}ak^2y_2 - aky_3.$$

Thus variables have been found such that

$$A = 2y_1'y_2 + ay_3^2, \quad cA - B = b_2y_2^2 + b_3y_3^2\ldots\ldots\ldots\ldots(4),$$

and the reduction is now complete. Geometrically, the distinction in (3) depends on whether $y_2 = 0$ is or is not one of the pair of lines $cA - B = 0$; when it is not, we take $y_3 = 0$ as the harmonic conjugate of $y_2 = 0$ with respect to this pair of lines. Then $y_1' = 0$ is the tangent to $A = 0$ at its second intersection with y_3, as in Fig. 4 †.

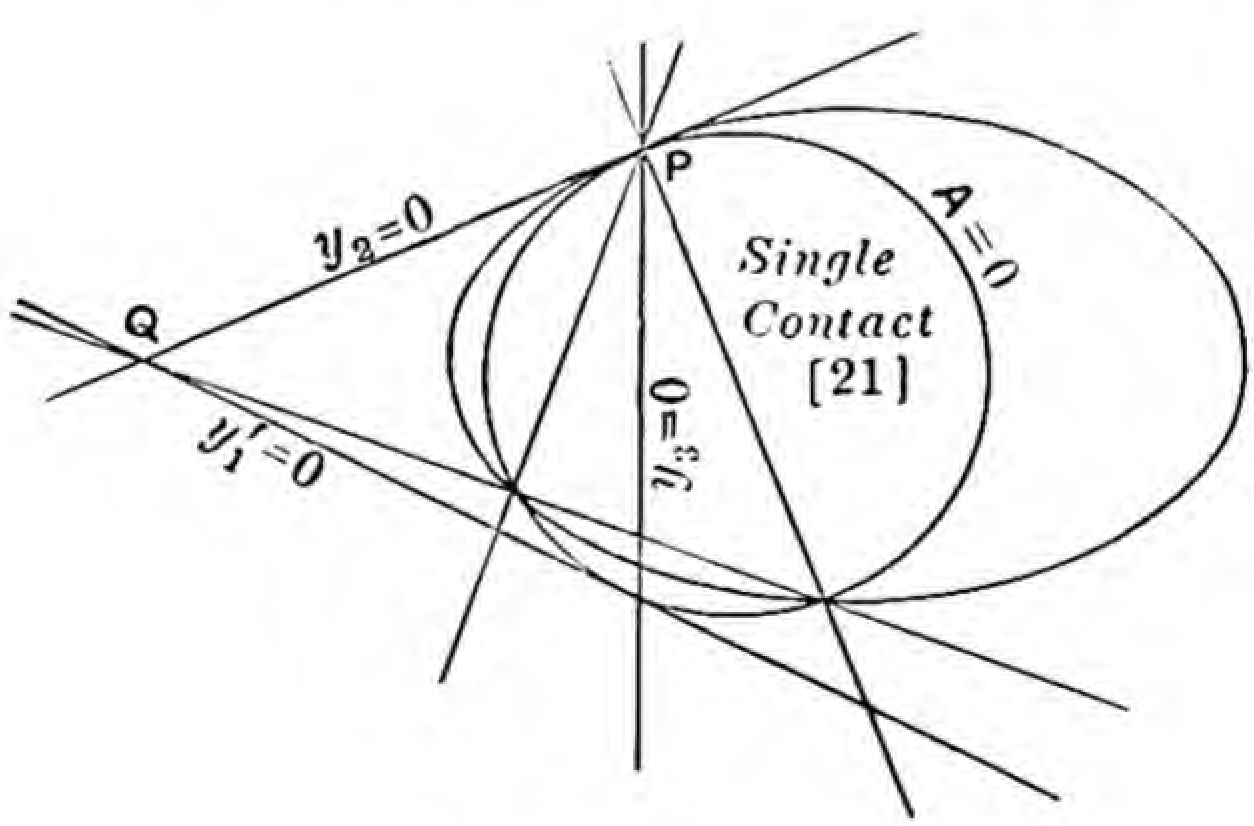


Fig. 4.

If, however, x_3^2 is not present in $(cA - B)$ *in* (3), we may write

$$cA - B = 2y_2(bx_3 + ky_2) = 2y_2y_3,$$

* Here the method is not really different from method (i): with more variables the two methods are quite distinct (see note † on p. 41).

† It will be seen that we are dealing with the same case as the second on p. 32; but from a different point of view.

so that $$A = 2y_1'y_2 + ay_3^2/b^2,$$
where $$y_1' = y_1 + \tfrac{1}{2}(ak^2/b^2)\,y_2 - (ak/b)\,y_3.$$
Thus the reduced forms are
$$A = 2y_1'y_2 + \alpha y_3^2, \quad cA - B = 2y_2y_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Here, from the geometrical point of view, $y_2 = 0$ is one of the two lines $cA - B = 0$ and $y_3 = 0$ is taken to be the other; $y_1' = 0$ is determined as before. The *conics will be seen to osculate* at* P, as in Fig. 5.

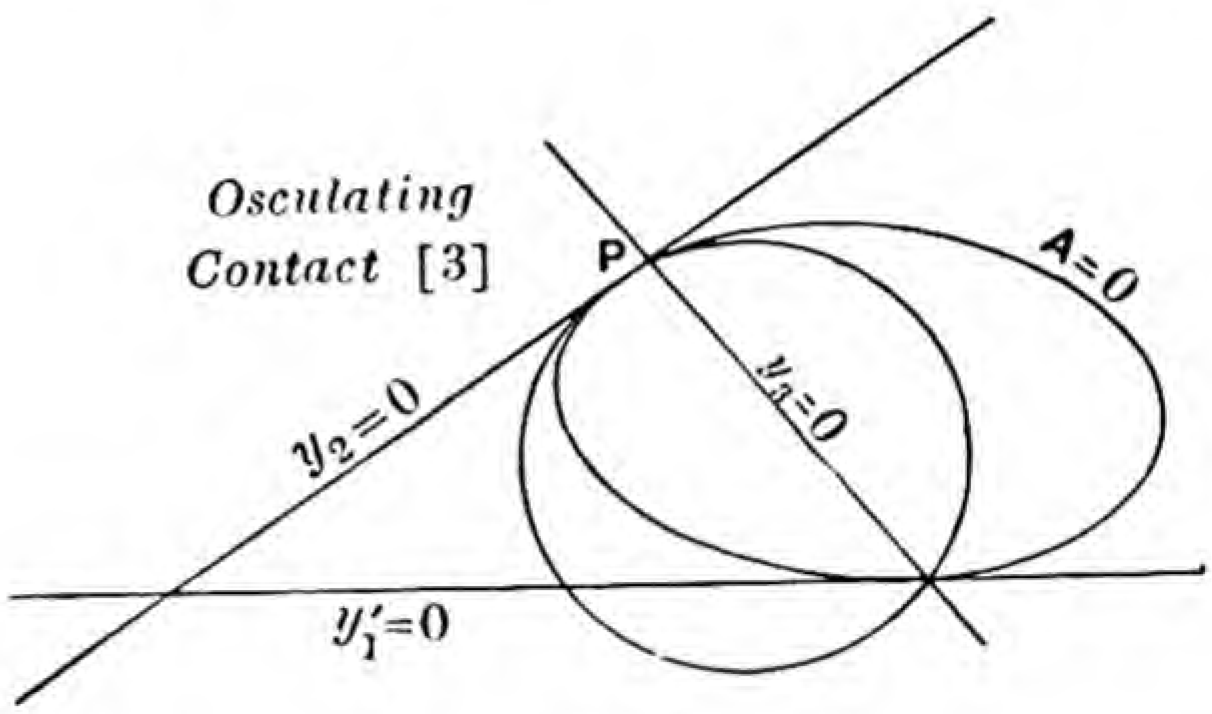


Fig. 5.

We note here that if $a = 0$ in equation (5), $\lambda A - B = 0$ always represents a pair of straight lines (see § 4 below).

§ 2. Secondly, suppose that the form $(cA - B)$ contains only one variable (x_3) so that $(\lambda - c)$ is a factor of every first minor in $|\lambda A - B|$; then
$$cA - B = kx_3^2;$$
choose if necessary two new variables x_1, x_2 so that the terms in A which are independent of x_3 reduce to the form† $ax_1^2 + bx_2^2$. If neither a nor b is zero, we can apply method (i) of Art. 7 to absorb the terms x_1x_3 and x_2x_3 in A; and then finally
$$A = y_1^2/a + y_2^2/b + lx_3^2, \qquad cA - B = kx_3^2 \quad \ldots\ldots\ldots\ldots(6).$$

But if (say) b is zero, the reduced forms are
$$A = y_1^2/a + 2y_2x_3, \qquad cA - B = kx_3^2 \quad \ldots\ldots\ldots\ldots\ldots(7).$$

* For, since y_1' cannot be zero in the neighbourhood of P, we may write
$$t = y_3/y_1', \quad u = y_2/y_1'.$$
Then the two conics are given by
$$(A)\quad u = -\tfrac{1}{2}at^2, \qquad (B)\quad u = -\tfrac{1}{2}act^2/(c+t).$$
Thus for $t = 0$, u, $\dfrac{du}{dt}$, $\dfrac{d^2u}{dt^2}$ are the same for the two conics; and this fact indicates osculation at $t = 0$.

† See Art. 8 for the methods of procedure.

Of course a, b may both be zero: but then we can write

$$A = 2y_2x_3, \quad cA - B = kx_3^2,$$

and so the family of forms contains only *two* independent variables.

From the geometrical point of view it is clear that when $(cA - B)$ is represented by a pair of coincident lines, the only cases possible are given by (i) double contact, (ii) four-point contact as in Figs 6, 7:—

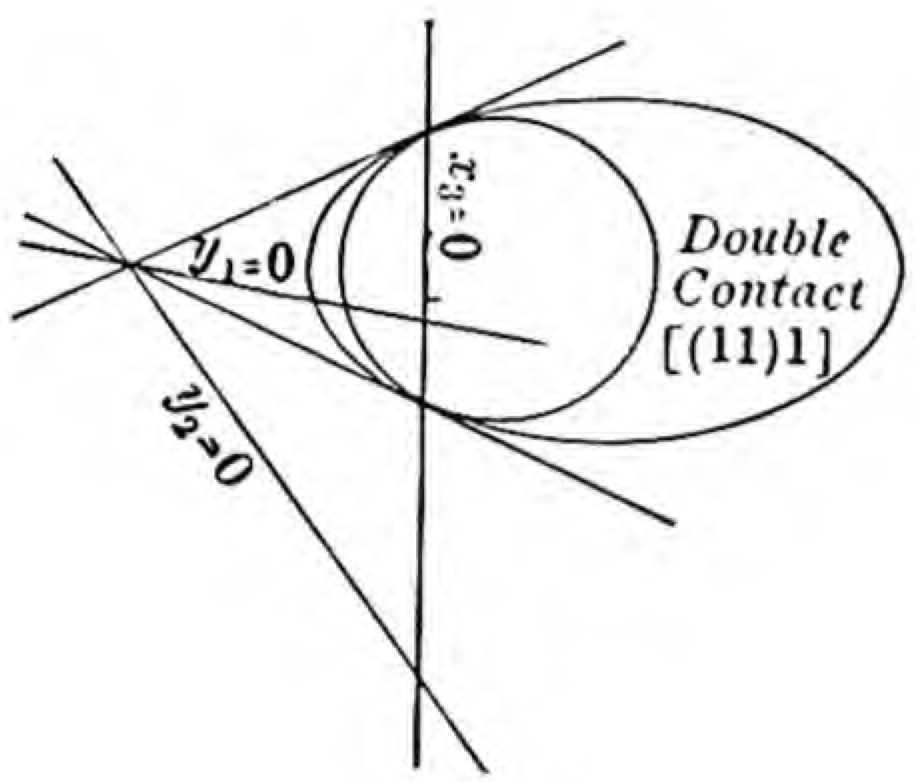


Fig. 6.

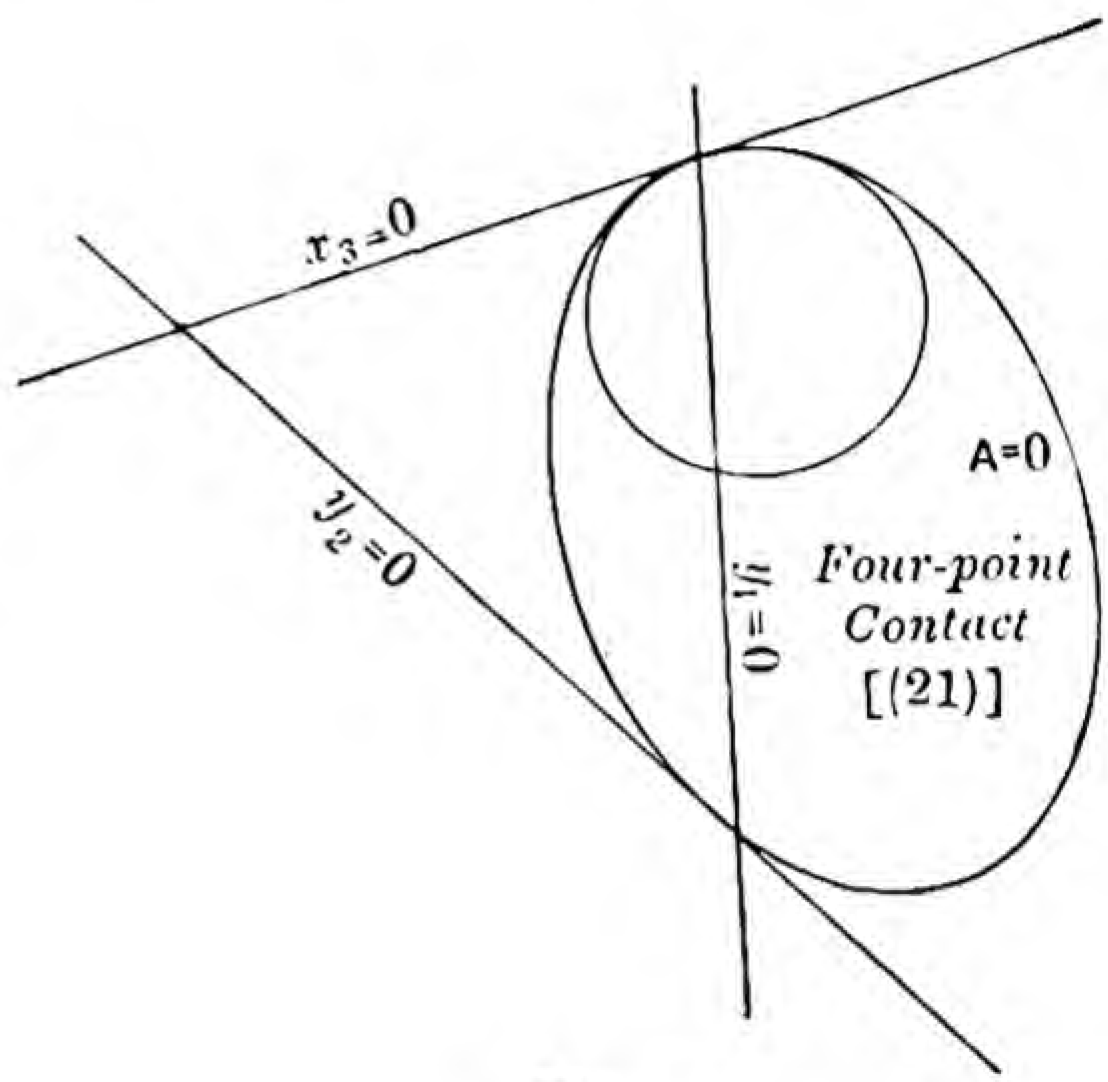


Fig. 7.

§ 3. Thirdly, $(cA - B)$ may be identically zero and then if A is brought to a sum of squares by Art. 8, B will be reduced at the same time.

§ 4. In addition to the cases already discussed we must examine *the singular case* when $|\lambda A - B|$ is zero for all values of λ. Here of course the determinants $|A|$ and $|B|$ are zero, and therefore the variables x_1, x_2, x_3 can be chosen so that x_1 does not occur in B nor x_3 in A; write therefore

$$A = ax_1^2 + 2bx_1x_2 + cx_2^2, \quad B = px_2^2 + 2qx_2x_3 + rx_3^2.$$

Then $|\lambda A - B| = \lambda a(pr - q^2) - \lambda^2 r(ac - b^2)$ so that a, r must be zero

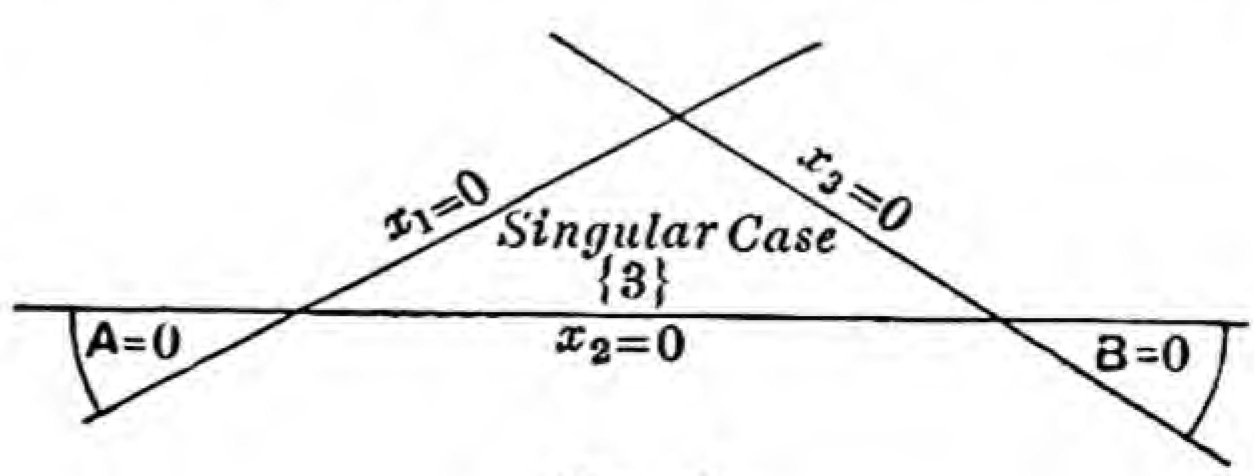


Fig. 8.

unless A and B are squares, a case already examined under § 2. Thus x_2 must be a factor of $(\lambda A - B)$ for all values of λ, and we may write

$$A = 2x_1x_2, \quad B = 2x_2x_3 \text{ .... (8).}$$

From the geometrical point of view it is clear that two pairs of straight lines can only give rise to a family of pairs of lines if one line is common to the two given pairs, as in Fig. 8.

14. We are now in a position to write out a **classified list of all the possible types of reduced forms in three variables.**

(i)
$$\left.\begin{aligned} A &= k_1x_1^2 + k_2x_2^2 + k_3x_3^2 \\ B &= k_1c_1x_1^2 + k_2c_2x_2^2 + k_3c_3x_2^2 \end{aligned}\right\} \text{ .... (1).}$$

Here the determinant $|\lambda A - B|$ has the three different factors $(\lambda - c_1)$, $(\lambda - c_2)$, $(\lambda - c_3)$; and so the indices of the invariant-factors are indicated by the symbol [111].

(ii) If $c_2 = c_1$ in the type (1), the determinant has factors $(\lambda - c_1)^2$, $(\lambda - c_3)$, and the first minors are *all* divisible by $(\lambda - c_1)$, so that the invariant-factors are $(\lambda - c_1)$, $(\lambda - c_1)$, $(\lambda - c_3)$; this is indicated by the symbol [(11) 1].

(iii) If in equation (1) $c_1 = c_2 = c_3$, the determinant has the cubed factor $(\lambda - c_1)^3$, every first minor is divisible by $(\lambda - c_1)^2$, and every element by $(\lambda - c_1)$; so that the invariant-factors are $(\lambda - c_1)$, $(\lambda - c_1)$, $(\lambda - c_1)$: which is indicated by [(111)].

(iv)
$$\left.\begin{aligned} A &= 2x_1x_2 + k_3x_3^2 \\ B &= 2c_1x_1x_2 + k_2x_2^2 + k_3c_3x_3^2 \end{aligned}\right\} \text{ .... (2).}$$

Here the determinant $|\lambda A - B|$ has the factors $(\lambda - c_1)^2$, $(\lambda - c_3)$, as in (ii): but one first minor is *not* divisible by $(\lambda - c_1)$, so that the invariant-factors are $(\lambda - c_1)^2$, $(\lambda - c_3)$: which is indicated by [21].

(v) If $c_3 = c_1$ in the type (2), the determinant contains the cubed factor $(\lambda - c_1)^3$, as in (iii); but one first minor is divisible only by $(\lambda - c_1)$, *not* by $(\lambda - c_1)^2$; thus the invariant-factors are $(\lambda - c_1)^2$, $(\lambda - c_1)$, indicated by [(21)].

(vi)
$$\left.\begin{aligned} A &= 2x_1x_2 + k_3x_3^2 \\ B &= 2c_1x_1x_2 + k_3c_1x_3^2 + 2x_2x_3 \end{aligned}\right\} \text{ .... (3).}$$

Here the determinant $|\lambda A - B|$ is divisible by $(\lambda - c_1)^3$ as in (iii) and (v); but there is one first minor which is not divisible by $(\lambda - c_1)$; so that there is but one invariant-factor $(\lambda - c_1)^3$, indicated by [3].

(vii)
$$A = 2x_1x_2, \quad B = 2x_2x_3 \text{ .... (4).}$$

Here $|\lambda A - B|$ is identically zero.

The reader will probably observe that we have given a list of 7 cases only, while there appear to have been 9 cases discussed in explaining the various methods of reduction (Art. 13). This difference arises from the fact that two of the cases may present themselves under different heads. For instance, if we have case (ii)

$$A = x_1^2 + x_2^2 + x_3^2, \quad B = c_1(x_1^2 + x_2^2) + c_3 x_3^2,$$

we may take either $(c_1 A - B)$ or $(c_3 A - B)$ for our discussion; the former contains only *one* variable, whereas the latter contains *two*. Thus the one type of reduced forms will present itself under two heads, when we consider the different methods of reduction*; geometrically, the fact is obvious on comparing Figs. 3 and 6.

Similarly in case (iv)

$$A = 2x_1 x_2 + x_3^2, \quad B = 2c_1 x_1 x_2 + x_2^2 + c_3 x_3^2,$$

if we consider $(c_1 A - B)$ and A, we find that $(c_1 A - B)$ contains x_2, x_3 but that A does not contain x_1^2; on the other hand, if we take $(c_3 A - B)$, this form contains x_1, x_2 and A does contain x_3^2. So that the pair of forms can be reduced in two different ways; to realise the geometrical equivalence, we need only compare Figs. 2 and 4.

15. It is a fundamental fact, to be noticed carefully, that **a knowledge of the invariant-factors of $|\lambda A - B|$ is sufficient to enable us to write down the type to which the forms A, B can be reduced, without actually carrying out the reduction.**

The truth of this statement will be recognized on observing that to every one of the types (i)—(vi) in Art. 14, corresponds one and only one set of invariant-factors. That the statement is true for quadratic forms in any number of variables will appear from Art. 20. There is, of course, one exception; in the type (vii) there are no invariant-factors. For this reason, the adjective *singular* is usually applied to this case; the necessary invariant is found in Art. 21.

For convenience of reference we give a table of the reduced forms, together with the geometrical relations of the corresponding conics (see p. 38).

* Incidentally it should be noted that here the reduction is not unique; because if $x_1 = y_1 \cos\theta - y_2 \sin\theta, \quad x_2 = y_1 \sin\theta + y_2 \cos\theta,$
we have $$x_1^2 + x_2^2 = y_1^2 + y_2^2.$$

And, generally if we write

$$x_1 = p_1 \xi_1 + p_2 \xi_2, \quad x_2 = q_1 \xi_1 + q_2 \xi_2,$$

we shall have $$k_1 x_1^2 + k_2 x_2^2 = l_1 \xi_1^2 + l_2 \xi_2^2,$$
provided only that $$k_1 p_1 p_2 + k_2 q_1 q_2 = 0.$$

Characteristic	Reduced forms	Geometrical meaning of $\lambda A - B = 0$
[111]	$A = k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2$ $B = c_1 k_1 x_1^2 + c_2 k_2 x_2^2 + c_3 k_3 x_3^2$	A family of conics with a definite self-polar triangle
[(11) 1]	$A = k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2$ $B = c_1 (k_1 x_1^2 + k_2 x_2^2) + c_3 k_3 x_3^2$	A family of conics having double contact: the self-polar triangle exists but is not unique
[(111)]	$A = k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2$ $B = c_1 (k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2)$	A family of coincident conics
[21]	$A = 2x_1 x_2 + k_3 x_3^2$ $B = 2c_1 x_1 x_2 + k_2 x_2^2 + c_3 k_3 x_3^2$	A family of conics having single contact: there is no proper self-polar triangle
[(21)]	$A = 2x_1 x_2 + k_3 x_3^2$ $B = c_1 (2x_1 x_2 + k_3 x_3^2) + k_2 x_2^2$	A family of conics with four-point contact
[3]	$A = 2x_1 x_2 + k_3 x_3^2$ $B = c_1 (2x_1 x_2 + k_3 x_3^2) + 2x_2 x_3$	A family of osculating conics
{3} Singular case	$A = 2x_1 x_2$ $B = 2x_2 x_3$	A family of pairs of straight lines: one line is common to all members of the family, the other line passes through a fixed point

With regard to this table it may be noted that the symbol {3} is explained in Art 21; and that it is now very easy to distinguish algebraically between conics which have single contact and conics which have double contact. In the older invariant-theory of conics (Salmon, *Conics*, Art. 378a) this distinction is not very simple.

16. Examples of reduction in three variables.

Example 1. *Type* [111]. Take the two quadratic forms*

$$A = 3x^2 - y^2 - z^2 - 6zx - 2xy,$$
$$B = 6x^2 - y^2 + 2z^2 + 6yz + 4xy.$$

The determinantal equation $|\lambda A - B| = 0$ gives

$$(\lambda - 1)(\lambda + 2)(13\lambda - 37) = 0.$$

We shall therefore work with the two forms $(B - A)$, $(2A + B)$, each of which will depend on two variables only (by Art. 5). Now

$$B - A = 3(x^2 + z^2 + 2yz + 2zx + 2xy) = 3(x + z)(x + 2y + z),$$

* *Mathematical Tripos*, 1892.

so write $t=x+z$, and then

$$B-A=3t(2y+t),$$

where y, t replace x_2, x_3 of Art. 13, equation (1), and x will take the place of x_1. When the other form $(2A+B)$ is expressed in terms of x, y, t, we have

$$\tfrac{1}{3}(2A+B)=4x^2-y^2+(2y-4x)(t-x)$$
$$=8x^2-2xy-4tx-y^2+2ty.$$

Next, we must isolate the terms which contain x, using Art. 7 (i); it will be seen that

$$\tfrac{1}{3}(2A+B)=\tfrac{1}{8}(8x-y-2t)^2-\tfrac{1}{8}(2t-3y)^2,$$

so that
$$2A+B=\tfrac{3}{8}(\xi^2-\eta^2),$$

where
$$\xi=8x-y\ \ -2t=6x-y-2z,$$
$$\eta=\quad -3y+2t=2x-3y+2z.$$

The variable ξ corresponds to y_1 of Art. 13, equation (2); but in the work of that article, η would not appear until a later stage. Now replace y in $(B-A)$ by its value $\tfrac{1}{3}(2t-\eta)$; the forms then become

$$B-A=t(7t-2\eta),\quad 2A+B=\tfrac{3}{8}(\xi^2-\eta^2).$$

The further reduction depends on that of the two forms $t(7t-2\eta)$ and $-\tfrac{3}{8}\eta^2$; and, since t^2 is present in the former, we can write

$$t(7t-2\eta)=\tfrac{1}{7}(\zeta^2-\eta^2),$$

where
$$\zeta=7t-\eta=5x+3y+5z.$$

Thus
$$B-A=\tfrac{1}{7}(\zeta^2-\eta^2),\quad 2A+B=\tfrac{3}{8}(\xi^2-\eta^2);$$

and it follows at once that $A=\tfrac{1}{8}\xi^2-\tfrac{13}{168}\eta^2-\tfrac{1}{21}\zeta^2,$

$$B=\tfrac{1}{8}\xi^2-\tfrac{37}{168}\eta^2+\tfrac{2}{21}\zeta^2.$$

The accuracy of our work is now checked by observing that the third factor of $|\lambda A-B|$ is evidently $(13\lambda-37)$, as it ought to be. Referring briefly to the geometrical interpretation of this example, we observe that the pair of lines $B-A=0$ meet in the point P (1, 0, -1). Then the polar of P with respect to either conic is $6x-y-2z=0$, or $\xi=0$ in the notation used above. Next, the second pair of lines $2A+B=0$ meet in the point Q (1, 2, 2) and the polar of Q is $5x+3y+5z=0$, or $\zeta=0$ in the foregoing notation. Finally PQ is $2x-3y+2z=0$, or $\eta=0$.

Example 2. *Type* $\lfloor 3\rfloor$. The two quadratic forms

$$A=2x^2+2y^2\qquad -2yz-2zx,$$
$$B=\ x^2+3y^2+z^2-4yz-2zx;$$

give a determinant $|\lambda A-B|$ which has the one invariant-factor $(\lambda-1)^3$.

Here $A-B=x^2-(y-z)^2$, so that we may take $y=z+t$, and then we have, corresponding to equation (1) of Art. 13,

$$A-B=x^2-t^2,\quad A=2z(t-x)+2t^2+2x^2.$$

Since z is the variable not present in $(A-B)$, and since z^2 does not occur in A, we take (see equation (2), Art. 7)

$$z'=z+t+x, \quad t'=t-x,$$

giving
$$A-B=-t'(t'+2x), \quad A=2z't'+4x^2,$$

as in equation (3) Art. 13.

Since x^2 is not present in $(A-B)$, we write $x'=2x+t'$, and then

$$A-B=-x't', \quad A=t'(2z'-2x'+t')+x'^2.$$

Thus the reduced forms are given by

$$A-B=\eta\zeta, \quad A=-\xi\eta+\zeta^2,$$

where
$$\begin{aligned} \xi &= 2z'-2x'+t' = -x+y+z, \\ \eta &= \qquad\quad -t' = \ \ x-y+z, \\ \zeta &= \qquad\ \ x' \qquad = x+y-z. \end{aligned}$$

Looking at this example geometrically we observe that the intersection of the pair of lines $A-B=0$ is the point $(0, 1, 1)$; and the polar of this point with respect to the conics is the line $-x+y-z=0$ or $\eta=0$. The line $\eta=0$ is then found to be one of the pair $A-B=0$, and so we take the other line $-x-y+z=0$ as $\zeta=0$.

Example 3. As exercises for reduction, the reader may take the following:—

(i) $A=3x^2+9y^2+4yz-2zx-6xy, \qquad B=5x^2+8y^2-2z^2-6zx-14xy,$

(ii) $A=5x^2+3y^2+2z^2+4yz-2zx+2xy,$
$$B=9x^2-3y^2-4z^2-8yz-10zx-6xy,$$

(iii) $A=5x^2-5y^2+z^2+6yz+10zx-4xy,$
$$B=10x^2+2y^2+10z^2-10yz+24zx-16xy.$$

The invariant-factors of $|\lambda A-B|$ are respectively

(i) $(\lambda-1)(\lambda-2)(\lambda-3)$, (ii) $(\lambda-1)(\lambda-5)(\lambda+9)$, (iii) $(\lambda-1)^2(\lambda-2)$.

The reduced form of (i) may be found in Salmon's *Conic Sections* (Art. 381, Ex. 5); and that of (ii) in Fiedler's translation (Art. 371, B. 1).

17. Reduction of two quadratic forms in four variables.

We shall here again introduce geometrical explanations to illustrate the algebra, which is very similar to the work for three variables in Art. 13. The variables x_1, x_2, x_3, x_4 will be taken as homogeneous coordinates in space, so that the equations $A=0$, $B=0$ represent two quadric surfaces.

Let $(\lambda-c)$ be a factor of the determinant $|\lambda A-B|$, then $(cA-B)$ depends on three independent variables at most; choose the variables x_1, x_2, x_3, x_4, so that x_1 does not occur in $(cA-B)$.

§ 1. First suppose that $(cA - B)$ contains three variables x_2, x_3, x_4; then, *if x_1^2 is present in A* we transform by method (i) of Art. 7 and so obtain

$$A = y_1^2/a_{11} + g(x_2, x_3, x_4), \quad cA - B = f(x_2, x_3, x_4) \ldots\ldots (1),$$

where

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4.$$

The reduction can now be completed by treating f and g according to the process of Art. 13. In this case the determinant $|\lambda A - B|$ has the invariant-factor $(\lambda - c)$, in addition to those of $|(\lambda - c) g + f|$. There are thus six ordinary cases and one singular case. Geometrically speaking, we find first a cone $cA - B = 0$ with its vertex at **P**, say; then if **P** is not on the quadric $A = 0$, the polar plane ϖ of **P** (which is the same for all the quadrics $\lambda A - B = 0$) does not pass through **P**. Thus **P** and ϖ may be taken as a vertex and the opposite face of the tetrahedron of reference; and the reduction depends on that of the family of conics in which ϖ cuts the quadrics.

But if x_1^2 is not present in A, we use Art. 7, equation (2) and obtain*

$$A = 2y_1y_2 + g(x_3, x_4)$$

where

$$a_{12}y_2 = a_{12}x_2 + a_{13}x_3 + a_{14}x_4$$

and y_1 is a linear function of x_1, x_2, x_3, x_4. Substitute now

$$x_2 = y_2 - (a_{13}x_3 + a_{14}x_4)/a_{12}$$

in $(cA - B)$ and we obtain say $h(y_2, x_3, x_4)$; to this we must apply method (iii) or (iv) of Art. 7, keeping y_2 unaltered.

If $h(0, x_3, x_4)$ is not a square, equation (3) of Art. 7 applies†, and we get

$$cA - B = ky_2^2 + h(0, y_3, y_4),$$

where $(y_3 - x_3)$ and $(y_4 - x_4)$ are simply multiples of y_2. Substitute in A for x_3, x_4 in terms of y_2, y_3, y_4 and then

$$g(x_3, x_4) = g(y_3, y_4) + 2y_2Y$$

where Y is a linear function of y_2, y_3, y_4.

Thus, if $z_1 = y_1 + Y$, we have

$$A = 2z_1y_2 + g(y_3, y_4), \quad cA - B = ky_2^2 + h(0, y_3, y_4) \ldots\ldots (2),$$

and the reduction is now completed by applying Arts. 2, 3. There are three cases in all and $|\lambda A - B|$ has the invariant-factor $(\lambda - c)^2$ as well as those of $|(\lambda - c) g + h|$. From the geometrical point of view the

* We assume that a_{12} is not zero; compare note on p. 32.

† Note that method (iii) is here quite different from method (i); this is one point of difference between the work for three and for four variables.

present case is distinguished by the fact that **P**, *the vertex of the cone* $cA - B = 0$, *is on the quadric* $A = 0$, so that the polar plane ϖ reduces to the common tangent-plane of the quadrics at **P**; and further *the plane* ϖ *does not touch the cone* $cA - B = 0$. In the analysis ϖ is taken as $y_2 = 0$; and then the polar line (l) of ϖ with respect to the cone is taken as the edge $y_3 = 0$, $y_4 = 0$. The line l cuts the quadric $A = 0$ again in **Q**, and the tangent-plane at **Q** is $z_1 = 0$; the final determination of the tetrahedron is given by the involution intercepted on the line $z_1 = 0$, $y_2 = 0$, by the system of quadrics.

When $h(0, x_3, x_4)$ *is a square*, we can apply equation (4) of Art. 7: but in practical work it saves trouble to proceed on slightly different lines. Suppose that the square is ly_4^2, say, where y_4 is linear in x_3 and x_4; thus we can write, using y_4 in place of x_4,

$$A = 2y_1y_2 + g_1(x_3, y_4), \quad cA - B = 2y_2\zeta + ly_4^2 \quad\ldots\ldots\ldots(3_1),$$

where ζ is linear in y_2, x_3, y_4.

Consider now g_1, this form will be expressible in one of the two types

$$\left.\begin{array}{ll} \textit{either} & g_1(x_3, y_4) = k_3y_3^2 + k_4y_4^2 \\ \textit{or} & g_1(x_3, y_4) = 2y_3y_4 \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(3_2),$$

according as y_4 is not or is a factor of g_1; in both cases y_3 is linear in x_3 and y_4. Thus we find, on introducing y_3 instead of x_3,

$$cA - B = 2y_2(\alpha y_2 + \beta y_3 + \gamma y_4) + ly_4^2,$$

where α, β, γ are constant coefficients. Thus

$$cA - B = 2y_2z_3 + lz_4^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3_3),$$

if

$$z_3 = \beta y_3 + (\alpha - \tfrac{1}{2}\gamma^2/l)\,y_2, \quad z_4 = y_4 + (\gamma/l)\,y_2.$$

Now β is not zero, since we suppose that $(cA - B)$ contains three independent variables: thus we may substitute z_3, z_4 instead of y_3, y_4 in the form A. It is then obvious from (3_2) that $g_1(x_3, y_4)$ takes one of the two shapes

$$\left.\begin{array}{ll} \textit{either} & g_1(x_3, y_4) = (k_3/\beta^2)\,z_3^2 + k_4z_4^2 + 2y_2Y \\ \textit{or} & g_1(x_3, y_4) = (2/\beta)\,z_3z_4 + 2y_2Y \end{array}\right\} \quad\ldots\ldots\ldots\ldots(3_4),$$

where, in each case, Y is linear in y_2, y_3, y_4. Thus, by writing z_1 in place of $(y_1 + Y)$, it is plain from (3_1), (3_3) and (3_4), that we have

$$\textit{either} \quad A = 2z_1y_2 + k_3'z_3^2 + k_4z_4^2, \quad cA - B = 2y_2z_3 + lz_4^2 \quad\ldots\ldots(4_1),$$

$$\textit{or} \quad A = 2z_1y_2 + 2z_3z_4, \quad cA - B = 2y_2z_3 + l'z_4^2 \quad\ldots\ldots(4_2),$$

where in (4_2) we have put z_4 in place of z_4/β. The family (4_1) has the two invariant-factors $(\lambda - c)^3$, $(\lambda - c)k_4 + l$; while (4_2) has the single invariant-factor $(\lambda - c)^4$.

In the two cases just examined, the plane ϖ ($y_2 = 0$) touches the cone $cA - B = 0$ along a generator l ($y_2 = 0$, $y_4 = 0$) : and the distinction between (4_1) and (4_2) turns on whether l is not or is a generator* of the quadric $A = 0$.

In the former case take m as a line through **P** which is harmonically conjugate to l with respect to the two generators at **P**; in the second case take m as the other generator at **P**. Then l is $y_2 = 0$, $z_4 = 0$ and m is $y_2 = 0$, $z_3 = 0$ in the analysis. To fix the planes of reference, draw the second tangent plane to the cone through the line m, so as to touch the cone along a generator n. Then the tangent-plane mn is $z_3 = 0$, and nl is $z_4 = 0$; while $z_1 = 0$ touches the quadric $A = 0$ at its second intersection with the line n.

§ 2. Secondly, suppose that $(cA - B)$ contains only the two variables x_3, x_4, so that $(\lambda - c)$ is a factor of every first minor of $|\lambda A - B|$ (Art. 5); and then the terms independent of x_3, x_4 in A can be transformed to the shape $ax_1^2 + bx_2^2$ by Art. 8; where of course one or both of a, b may be zero.

If neither a nor b is zero, we can introduce new variables y_1, y_2 (by method (i) of Art. 7) to absorb all such products as x_1x_3, x_2x_3, ... in A; and then we arrive at the forms

$$cA - B = f(x_3, x_4), \quad A = y_1^2/a + y_2^2/b + g(x_3, x_4) \ldots\ldots (5),$$

and the further reduction depends on that of f and g (by Arts. 2, 3). There are thus three cases, and each of them has the two invariant-factors $(\lambda - c)$, $(\lambda - c)$, in addition to the invariant-factors given by the determinant $|(\lambda - c)g + f|$.

In the geometrical interpretation, *the equation $cA - B = 0$ now represents a pair of planes*: let their line of intersection be denoted by l (which of course is the line $x_3 = 0$, $x_4 = 0$ of the analysis). The three cases just settled arise when *l is neither a tangent nor a generator of* $A = 0$; then take m, the polar line of l with respect to $A = 0$, and this is the line $x_1 = 0$, $x_2 = 0$. The remainder of the discussion depends on the involution intercepted on m by the family of quadrics $\lambda A - B = 0$; but the determination of the planes $y_1 = 0$, $y_2 = 0$ is not unique, as they may be any two planes through m which are conjugate with respect to $A = 0$.

Next suppose that *b is zero, a being different from zero*; then by applying method (i) of Art. 7 to x_1 and method (ii) to x_2, we find

$$A = y_1^2/a + 2y_2y_3 + kx_4^2, \quad cA - B = f_1(y_3, x_4) \ldots\ldots\ldots (6).$$

* It is always a tangent to the quadric, because ϖ is the tangent plane at **P**.

The rest of the work is the same as that of Art. 13, equations (3)—(5); and there are two cases according as x_4^2 is or is not present in $f_1(y_3, x_4)$. In the former case, the invariant-factors of $|\lambda A - B|$ are $(\lambda - c)^2$, $(\lambda - c)$ and a linear invariant-factor: in the latter they are $(\lambda - c)^3$ and $(\lambda - c)$.

From the geometrical point of view *the line l is here a tangent to the quadric* $A = 0$ and so also is its polar line m; and the line m is now taken as $y_1 = 0$, $y_3 = 0$, the plane $y_3 = 0$ being the tangent plane lm. The plane $y_1 = 0$ may be any plane through m (of course not coincident with lm); and the rest of the work reduces to considering the sections of the quadrics by $y_1 = 0$. The further subdivision simply turns on whether lm happens to be one of the two planes $cA - B = 0$ or not.

In the third place *if a, b are both zero* we can first choose the variables x_3, x_4 so that

$$cA - B = kx_3^2 + lx_4^2 \qquad \text{.......................} (7_1),$$

and then A at once takes the form

$$A = 2y_1x_3 + 2y_2x_4 \qquad \text{.......................} (7_2).$$

Thus $|\lambda A - B|$ has the invariant-factors $(\lambda - c)^2$, $(\lambda - c)^2$.

Here l is a generator of the quadric and we take $x_3 = 0$, $x_4 = 0$ as two planes harmonically conjugate with respect to the two planes $cA - B = 0$. These planes $x_3 = 0$, $x_4 = 0$ cut $A = 0$ in two more generators, and $y_1 = 0$, $y_2 = 0$ are any other tangent-planes of the quadric through these two generators.

§ 3. Thirdly, $(cA - B)$ may contain only one variable x_4, so that $(\lambda - c)$ is a factor of every second minor of $|\lambda A - B|$. Write then $cA - B = lx_4^2$ and let the terms of A which do not contain x_4 be reduced to the form

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2$$

by Art. 8. Then, *if a_1, a_2, a_3 are different from zero*, we can apply method (i) of Art. 7 to each of the variables x_1, x_2, x_3, and so obtain

$$A = y_1^2/a_1 + y_2^2/a_2 + y_3^2/a_3 + kx_4^2, \quad cA - B = lx_4^2 \qquad \text{.........} (8).$$

The invariant-factors are then $(\lambda - c)$, $(\lambda - c)$, $(\lambda - c)$ and $k(\lambda - c) + l$.

But *if a_3 is zero*, we get (by applying method (ii) of Art. 7 to x_3)

$$A = y_1^2/a_1 + y_2^2/a_2 + 2y_3x_4, \quad cA - B = lx_4^2 \qquad \text{...........} (9),$$

and then the invariant-factors are

$$(\lambda - c)^2, \quad (\lambda - c), \quad (\lambda - c).$$

It might be thought that two of a_1, a_2, a_3 could be zero; but if this happens, the family $(\lambda A - B)$ contains only three variables, and so does not belong to the present discussion.

In the geometrical representation, $cA - B = 0$ *now represents two coincident planes*; and the distinction between the two cases depends on whether this plane does not or does touch the quadric $A = 0$.

§ 4. No special discussion arises if $(cA - B)$ is identically zero; A can then be reduced by Art. 8 and B is automatically reduced at the same time.

§ 5. In dealing with *the singular case* we observe that, since $|A| = 0$, and $|B| = 0$ we can choose the variables so that x_1 does not occur in B, nor x_4 in A.

If x_1^2 *is present in* A we can apply method (i) of Art. 7, so that we write

$$A = y_1^2/a + f(x_2, x_3), \quad B = g(x_2, x_3, x_4).$$

Then, since $|\lambda A - B|$ is identically zero, we must have $|\lambda f - g|$ identically zero, and so (by Art. 13, § 4) the variables can be chosen so that

$$f(x_2, x_3) = 2y_2y_3, \quad g(x_2, x_3, x_4) = 2y_3y_4.$$

Hence $$A = y_1^2/a + 2y_2y_3, \quad B = 2y_3y_4 \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

A similar method applies if x_4^2 is present in B; so we shall *now suppose neither* x_1^2 *nor* x_4^2 *to be present in* $\lambda A - B$. If we apply method (ii) of Art. 7 to A, we obtain

$$A = 2y_1y_2 + kx_3^2, \quad B = g(y_2, x_3, x_4).$$

Now if $2m$ is the coefficient of x_3x_4 in g, it will be seen that

$$|\lambda A - B| = \lambda^2 m^2,$$

which can only vanish identically if $m = 0$. Then g contains only x_3^2 and terms divisible by y_2; thus we can introduce y_4 so that

$$A = 2y_1y_2 + kx_3^2, \quad B = 2y_2y_4 + lx_3^2 \ldots\ldots\ldots\ldots\ldots (11).$$

This completes the reduction: $|\lambda A - B|$ has the one invariant-factor $(\lambda k - l)$.

It is not easy to obtain an entirely satisfactory discussion of the singular case on geometrical lines. But it may be observed that we have already discussed all forms of the intersection of two cones except when the curve consists of two *coincident* generators and a plane conic. Here the cones touch along a common generator, and it is easy to see that this gives the geometrical interpretation of the reduced forms (10) or (11). Hence we may state the result:—

If the family $(\lambda A - B)$ *consists solely of cones (not having a common vertex), their vertices lie along a straight line, which is a common generator of all the cones; and the cones have a common tangent-plane along this generator.*

18. Table shewing the reduced forms in four variables.

Characteristic	Reduced forms	Geometrical form of the curve of intersection of the quadrics $\lambda A - B = 0$	Clebsch's number*
[1111]	$A = k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2 + k_4 x_4^2$ $B = c_1 k_1 x_1^2 + c_2 k_2 x_2^2 + c_3 k_3 x_3^2 + c_4 k_4 x_4^2$	A general twisted quartic of the first species	1
[(11) 11]	As in the previous case, with $c_2 = c_1$	Two conics intersecting in two distinct points P, Q (not necessarily real points): the quadrics touch at P, Q	6
[(11)(11)]	As in the first case, with $c_2 = c_1$, $c_4 = c_3$	Four generators intersecting in four points, at which the quadrics touch	9
[(111) 1]	As in the first case, with $c_3 = c_2 = c_1$	A single conic counted twice: the quadrics touch all along the conic (ring-contact)	12
[(1111)]	As in the first case, with $c_4 = c_3 = c_2 = c_1$	No proper intersection: all the quadrics coincide	Absent
[211]	$A = 2x_1 x_2 + k_3 x_3^2 + k_4 x_4^2$ $B = 2c_1 x_1 x_2 + k_1 x_1^2 + k_3 c_3 x_3^2 + k_4 c_4 x_4^2$	A nodal quartic: all the quadrics touch at the node	2
[(21) 1]	As in the previous case, with $c_3 = c_1$	Two conics which touch: all the quadrics have stationary contact at the point of contact	7
[2 (11)]	As in the sixth case, with $c_4 = c_3$	A conic and two generators, intersecting in three points, at which the quadrics touch	8
[(211)]	As in the sixth case, with $c_4 = c_3 = c_1$	Two generators counted twice: the quadrics touch all along the generators	13

* This column gives the number attached to the corresponding case in the list made by Lindemann in his edition of Clebsch's *Vorlesungen über Geometrie*, Bd. 2, pp. 215—237. It will be observed that Lindemann takes first all the cases in which no root of $|\lambda A - B|$ is also a root of all the first minors; that is, the cases without round brackets. He takes next those in which there are common roots of all the first minors, but not of the second minors; and so on. Sylvester adopts the same order in his paper (quoted on p. 27), except that 9 and 10 are interchanged.

Characteristic	Reduced forms	Geometrical form of the curve of intersection of the quadrics $\lambda A - B = 0$	Clebsch's number*
[22]	$A = 2x_1x_2 + 2x_3x_4$ $B = 2c_1x_1x_2 + k_1x_1^2 + 2c_3x_3x_4 + k_3x_3^2$	A generator and a twisted cubic: the generator cuts the cubic in two distinct points, at which the quadrics touch	4
[(22)]	As in the last case, with $c_3 = c_1$	Three generators, one counted twice: this generator intersects each of the other two	11
[31]	$A = 2x_1x_2 + k_3x_3^2 + k_4x_4^2$ $B = c_1(2x_1x_2 + k_3x_3^2) + c_4k_4x_4^2 + 2x_2x_3$	A cuspidal quartic; the quadrics have stationary contact at the cusp	3
[(31)]	As in the last case, with $c_4 = c_1$	Two generators and a conic which touches the plane of the generators at their intersection	10
[4]	$A = 2x_1x_2 + 2x_3x_4$ $B = 2c_1x_1x_2 + 2c_1x_3x_4 + 2x_2x_3 + k_4x_4^2$	A generator and a twisted cubic: the generator touches the cubic	5
[{3} 1]	$A = 2x_1x_2 + k_4x_4^2$ $B = 2x_2x_3 + k_4c_4x_4^2$	A generator counted twice and a conic. This is the singular case	14

For the theory of twisted quartics, the reader may consult Salmon's *Geometry of Three Dimensions* (Arts. 347—351); it is there proved that there are two species:—*Those which are the intersection of two quadrics* and *those through which only one quadric can pass*†.

On a quartic of type [1111], the coordinates are *not* rational but require elliptic functions for their expression. To see this, consider equation (1) of Art. 17 and let a generator be found common to the two cones $f=0$, $g=0$; take y_4 as the tangent-plane to f along this generator and y_3 as another plane through the generator. The equations to the quartic can then be written

$$y_2y_4 = y_3^2, \quad y_1^2 = h(y_2, y_3, y_4),$$

* See note on p. 46.

† On a quartic of the second species the coordinates are *rational*; thus we may choose the coordinates so that (*l.c.* Arts. 349, 351)

either $\quad x_1 = t^4 + a, \quad x_2 = t^3 + b, \quad x_3 = t^2 + c, \quad x_4 = t + d,$

or $\quad x_1 = t^4 + at^2, \quad x_2 = t^3, \quad x_3 = t, \quad x_4 = 1.$

For other theorems, see a paper by Forsyth, *Quarterly Journal of Mathematics*, vol. XXVII, 1895, p. 247.

where h does not contain y_2^2, but must contain $y_2 y_3$, because the two cones f, g cannot have a common tangent-plane y_4 along their common generator. Thus on the quartic we may write

$$y_2 = t^2, \quad y_3 = t, \quad y_4 = 1, \quad y_1^2 = F(t), \text{ a cubic in } t.$$

By writing $s = at + b$, we can find a and b so that the cubic $F(t)$ reduces to Weierstrass's type $(4s^3 - g_2 s - g_3)$. Then on the quartic we have

$$y_1 = \wp' u, \quad z_2 = \wp^2 u, \quad z_3 = \wp u, \quad y_4 = 1,$$

where $$z_2 = a^2 y_2 + 2ab y_3 + b y_4^2, \quad z_3 = a y_3 + b y_4.$$

To illustrate, take the reduced forms given in the table; then if we write

$$a_1 = (c_2 - c_3)(c_1 - c_4), \quad \xi_1 = x_1 [k_1/k_4 (c_2 - c_3)]^{\frac{1}{2}},$$

with symmetrical values for a_2, a_3 and ξ_2, ξ_3, the quartic satisfies

$$a_1 \xi_1^2 + a_2 \xi_2^2 + a_3 \xi_3^2 = 0, \quad (c_2 - c_3)\xi_1^2 + (c_3 - c_1)\xi_2^2 + (c_1 - c_2)\xi_3^2 + x_4^2 = 0,$$

x_4 taking the place of y_1 in the foregoing explanation. The common generators are given by $\xi_1^2 = \xi_2^2 = \xi_3^2$; and the tangent-plane to the first cone along the line $\xi_1 = \xi_2 = \xi_3$, is

$$a_1 \xi_1 + a_2 \xi_2 + a_3 \xi_3 = 0.$$

Thus we write $$t = [\beta(\xi_1 - \xi_2) + \gamma(\xi_2 - \xi_3)]/(a_1\xi_1 + a_2\xi_2 + a_3\xi_3),$$

and then the values of t corresponding to $x_4 = 0$ (or to $\xi_1^2 = \xi_2^2 = \xi_3^2$) are seen to be $t = \infty$, β/a_1, γ/a_2, $-(\beta+\gamma)/a_3$. Hence, to make the sum of the finite roots of t equal to zero, we take

$$\beta = a_1(a_2 - a_3), \quad \gamma = a_2(a_3 - a_1), \quad -(\beta + \gamma) = a_3(a_1 - a_2).$$

It will then be found, after some algebra, that we can write

$$\xi_1 = [t - (a_2 - a_3)]^2 + 9a_2 a_3, \text{ etc.}$$

$$x_4^2 = 12P[t - (a_2 - a_3)][t - (a_3 - a_1)][t - (a_1 - a_2)],$$

where $$P = -(c_1 a_1 + c_2 a_2 + c_3 a_3) = (c_2 - c_3)(c_3 - c_1)(c_1 - c_2).$$

Hence, finally, if $t = \rho \wp u$, $a_2 - a_3 = \rho e_1$, etc., we get

$$\frac{x_1}{x_4} = \left[\frac{\rho k_4}{3k_1(c_3 - c_1)(c_1 - c_2)}\right]^{\frac{1}{2}} \frac{(\wp u - e_1)^2 - (e_1 - e_2)(e_1 - e_3)}{\wp' u} \text{ etc.}$$

An alternative method is given by Burnside (*Messenger of Mathematics*, vol. XXIII, 1894, p. 89).

In contrast to this, the quartics of types [211] and [31] can be expressed rationally. We leave the following statements to the reader as easy exercises in reduction:—

On the *nodal quartic* [211], we can write

$$y_1 = t^4 + a, \quad y_2 = t^3, \quad y_3 = t^2, \quad y_4 = t.$$

On the *cuspidal quartic* [31], we can write

$$y_1=t^4,\quad y_2=t^2,\quad y_3=t,\quad y_4=1.$$

On the *twisted cubic*, [22] or [4], we can write

$$y_1=t^3,\quad y_2=t^2,\quad y_3=t,\quad y_4=1.$$

19. Examples of reduction in four variables.

Example 1. *Type* [4].

Let us take the two forms given by

$$A=-x_2^2+x_3^2-x_4^2-x_3x_1+x_1x_2+x_1x_4,$$
$$A+B=\tfrac{3}{4}x_2^2-\tfrac{1}{4}x_3^2-\tfrac{1}{4}x_4^2-\tfrac{1}{2}x_2x_3-\tfrac{1}{2}x_2x_4+\tfrac{3}{2}x_3x_4,$$

for which $|\lambda A-B|$ has the invariant factor $(\lambda+1)^4$. Since $A+B$ contains only x_2, x_3, x_4, the preliminary steps have been avoided; then, since x_1^2 is not present in A, we use equation (2) of Art. 7 and obtain

$$A=2y_1y_2+2x_3x_4-2x_4^2,\quad A+B=y_2(3y_2+2x_3-4x_4)+x_4^2,$$

where

$$y_1=x_1-x_2-x_3+x_4,\quad 2y_2=x_2-x_3+x_4.$$

It is plain that the forms are already reduced to the type (3_1) of Art. 17, x_4 taking the place of y_4. Now here x_4 is a factor of A, when $y_2=0$; so we write $y_3=x_3-x_4$, and then

$$A=2y_1y_2+2y_3x_4,\quad A+B=y_2(3y_2+2y_3-2x_4)+x_4^2.$$

It is now easy to reduce $(A+B)$ to the type (3_3) of Art. 17, thus

$$A+B=2y_2^2+2y_2y_3+(x_4-y_2)^2=2y_2z_3+y_4^2,$$

if

$$z_3=y_2+y_3,\quad y_4=x_4-y_2.$$

Then

$$A=2y_1y_2+2(z_3-y_2)(y_4+y_2)=2z_1y_2+2z_3y_4,$$

where

$$z_1=y_1-y_2+z_3-y_4.$$

We have thus completed the reduction. It will be seen that

$$\begin{aligned} z_1&=x_1-\tfrac{1}{2}x_2-\tfrac{1}{2}x_3-\tfrac{1}{2}x_4,\\ y_2&=\ \ \tfrac{1}{2}x_2-\tfrac{1}{2}x_3+\tfrac{1}{2}x_4,\\ z_3&=\ \ \tfrac{1}{2}x_2+\tfrac{1}{2}x_3-\tfrac{1}{2}x_4,\\ y_4&=-\tfrac{1}{2}x_2+\tfrac{1}{2}x_3+\tfrac{1}{2}x_4,\end{aligned}$$

is the reducing substitution.

In the geometrical interpretation the vertex $(1, 0, 0, 0)$ of the cone $A+B=0$ lies on the quadric $A=0$; and the tangent-plane to the quadric at this point is $x_2-x_3+x_4=0$, or $y_2=0$ in the notation above. On considering the cone it is found that $y_2=0$ touches the cone along the generator $y_2=0$, $x_4=0$, and further that this generator is a generator of the quadric. Thus we are to take, as two edges of the tetrahedron, the two generators of the quadric in the plane $y_2=0$; namely $y_2=0$, $x_4=0$ and $y_2=0$, $x_3-x_4=0$.

It is easily found that the second tangent-plane to the cone (drawn from the second of these generators) is $y_2+x_3-x_4=0$, or $z_3=0$. And the polar plane of the line $y_2=0$, $x_3-x_4=0$, with respect to the cone, is found to be $-x_2+x_3+x_4=0$, or $y_4=0$.

The three planes $y_2=0$, $z_3=0$, $y_4=0$ form three faces of the tetrahedron; to get the fourth face, consider the second intersection of the line $z_3=0, y_4=0$, with the quadric; this point is easily found to be $(1, 1, 0, 1)$, and the tangent-plane there is $2x_1-x_2-x_3-x_4=0$, or $z_1=0$.

We have thus completely determined the final tetrahedron of reference.

Example 2. *Type* [(22)].

Let us take the family of forms defined by

$$A=-x_2^2+x_3^2-x_4^2-x_3x_1+x_1x_2+x_1x_4,$$
$$A-B=(x_1-x_4)(x_1-x_2-x_3),$$

which give the invariant-factors $(\lambda-1)^2$, $(\lambda-1)^2$.

Here write $$\xi_3=x_3+x_2-x_1, \quad \xi_4=x_4-x_1,$$

and we have, on substitution for x_3, x_4,

$$A-B=\xi_3\xi_4, \quad A=\xi_3^2-\xi_4^2-2x_2\xi_3+x_1\xi_3-x_1\xi_4.$$

Since A is zero when $\xi_3=0$, $\xi_4=0$, we transform $(A-B)$ as in § 2 equation (7_1) of Art. 17, writing

$$y_3=\tfrac{1}{2}(\xi_3+\xi_4), \quad y_4=\tfrac{1}{2}(\xi_3-\xi_4)$$

so that $$A-B=y_3^2-y_4^2, \quad A=4y_3y_4-2x_2(y_3+y_4)+2x_1y_4.$$

Thus, with $$y_1=2y_4-x_2, \quad y_2=x_1-x_2,$$

we get $$A-B=y_3^2-y_4^2, \quad A=2y_1y_3+2y_2y_4.$$

Of course this reduction is not unique, for we might take in the first place

$$y_3=\tfrac{1}{2}(a\xi_3+b\xi_4), \quad y_4=\tfrac{1}{2}(a\xi_3-b\xi_4),$$

where a, b may have any values; and secondly, in transforming A the term y_3y_4 may be distributed arbitrarily between $y_1y_3+y_2y_4$.

In the geometrical representation the line of intersection of the planes $A-B=0$ is a generator of the quadric $A=0$; and the planes $y_3=0$, $y_4=0$ are any two planes through the generator which are harmonically conjugate with respect to the planes $A-B=0$. These two planes cut the surface in three generators; and to complete the tetrahedron of reference, any generator may be taken of the system to which the line $y_3=0$, $y_4=0$ belongs.

20. General method of reduction: n variables.

As before let $\lambda=c$ be one root of the determinantal equation $|\lambda A-B|=0$, then the variables $x_1, x_2, \ldots, x_n$ can be so chosen that $(cA-B)$ is independent of at least one variable, say x_1 (see Art. 5). We have then

$$cA-B=f(x_2, x_3, \ldots, x_n), \quad A=g(x_1, x_2, \ldots, x_n) \ldots\ldots(1).$$

If a_{11}, *the coefficient of* x_1^2 *in* A, *does not vanish*, by method (i) of Art. 7, we may write

$$A=y_1^2/a_{11}+h(x_2, x_3, \ldots, x_n)$$

where $$y_1=a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n.$$

Thus

$$\lambda A - B = (\lambda - c)\, y_1^2/a_{11} + [(\lambda - c)\, h\,(x_2, \ldots, x_n) + f(x_2, \ldots, x_n)] \ldots (2),$$

and so the reduction of $(\lambda A - B)$ depends on that of $[(\lambda - c)\, h + f]$ which contains only $(n-1)$ variables. We note that here the determinant $|\lambda A - B|$ has an invariant-factor $(\lambda - c)$ in addition to those of the determinant $|(\lambda - c)\, h + f|$.

In this way we proceed, step-by-step, until we arrive at a pair of forms containing only four variables, to which the results of the previous articles are at once applicable. Of course it may happen that at some stage in the process, the coefficient which corresponds to a_{11} in the above work is zero; and then the following methods will be used.

If x_1^2 *does not appear in* A, we apply method (ii) of Art 7, and then, assuming that $x_1 x_2$ appears in A, we find

$$A = 2y_1 y_2 + h_1\,(x_3, x_4, \ldots, x_n) \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where y_1 is a linear function of $x_1, x_2, \ldots, x_n$ and y_2 of $x_2, x_3, \ldots, x_n$. Substituting in $(cA - B)$ for x_2 in terms of $y_2, x_3, x_4, \ldots, x_n$, we have

$$cA - B = f_1\,(y_2, x_3, x_4, \ldots, x_n).$$

Apply method (iii) or (iv) of Art. 7 to this form and we shall have

either $$cA - B = ky_2^2 + f_1\,(0, y_3, \ldots, y_n) \ldots\ldots\ldots\ldots\ldots\ldots (4_1),$$

or $$cA - B = 2y_2 y_3 + f_1\,(0, 0, y_4, \ldots, y_n) \ldots\ldots\ldots\ldots\ldots (5_1).$$

In the former case $(y_3 - x_3), (y_4 - x_4), \ldots, (y_n - x_n)$ are multiples of y_2; and in the latter y_3 is a linear function of x_3 and y_2, while the differences $(y_4 - x_4), \ldots, (y_n - x_n)$ are linear combinations of y_2 and y_3.

Taking first equation (4_1) we substitute in A, $y_3, \ldots, y_n$ instead of $x_3, \ldots, x_n$ and then

$$A = 2y_1 y_2 + h_1\,(y_3, y_4, \ldots, y_n) + y_2 Y,$$

where Y is a linear function of $y_2, y_3, \ldots, y_n$. So we put $z_1 = y_1 + \frac{1}{2} Y$ and obtain

$$A = 2z_1 y_2 + h_1\,(y_3, y_4, \ldots, y_n),$$

so that $$\lambda A - B = 2\,(\lambda - c)\, z_1 y_2 + ky_2^2 + [(\lambda - c)\, h_1 + f_1] \ldots\ldots\ldots (4_2).$$

The form $(\lambda - c)\, h_1 + f_1$ depends on $(n-2)$ variables only, and we can start again to reduce this form. Here $|\lambda A - B|$ has an invariant-factor $(\lambda - c)^2$ as well as those of the determinant $|(\lambda - c)\, h_1 + f_1|$.

Turning next to equation (5_1) we are left with

$$\left.\begin{aligned} A &= 2z_1 y_2 + h_1\,(y_3, y_4, \ldots, y_n) \\ cA - B &= 2y_2 y_3 + f_2\,(y_4, \ldots, y_n) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (5_2).$$

Since h_1 contains one more variable than f_2, the methods given for

reducing f and g in equation (1) can be repeated, to reduce h_1 and f_2. This reduction will change $y_3, y_4, \ldots, y_n$, but the only change which concerns us at present is that of y_3; thus, suppose that $y_3 = \xi_3 + \sigma$, where σ is linear in $y_4, y_5, \ldots, y_n$. Inserting this value for y_3, we find

$$cA - B = 2y_2\xi_3 + 2y_2\sigma + f_2(y_4, \ldots, y_n),$$

so that, since the determinant of f_2 is not zero*, we may apply method (iii) of Art. 7 to the form $2y_2\sigma + f_2(y_4, \ldots, y_n)$ and we obtain

$$cA - B = 2y_2(\xi_3 + ky_2) + f_2(z_4, \ldots, z_n) = 2y_2z_3 + f_2(z_4, \ldots, z_n),$$

where $(z_3 - \xi_3), (z_4 - y_4), \ldots, (z_n - y_n)$ are multiples of y_2. Then A takes the form

$$A = 2z_1y_2 + h_2(\xi_3, y_4, \ldots, y_n) = 2z_1y_2 + h_2(z_3, z_4, \ldots, z_n) + y_2\sigma',$$

where σ' will be linear in $y_2, z_3, z_4, \ldots, z_n$. Hence, if $w_1 = z_1 + \frac{1}{2}\sigma'$, we have

$$A = 2w_1y_2 + h_2(z_3, z_4, \ldots, z_n),$$
$$cA - B = 2y_2z_3 + f_2(z_4, \ldots, z_n).$$

In actual work, however, it is usually better not to apply method (iv) and to avoid introducing $y_3, y_4, \ldots, y_n$ as in equation (5_1). We should start from (3), and write instead of (5_2)

$$A = 2y_1y_2 + h(x_3, x_4, \ldots, x_n), \quad cA - B = 2y_2\zeta + f_1(0, x_3, x_4, \ldots, x_n) \ldots (5_3),$$

where ζ is linear in $y_2, x_3, \ldots, x_n$. We can proceed to reduce h and f_1 (since the determinant of the latter is zero) by the methods used in equation (1) above. To explain how to complete the work, suppose that in an example we found

$$A = 2y_1y_2 + 2y_3y_4 + y_5^2 + y_6^2, \quad cA - B = 2y_2\zeta + 2y_4y_5 + y_6^2,$$

where $\zeta = y_2 + y_3 + y_4 + y_5 + y_6$. We should then write

$$z_4 = y_4 + y_2, \quad z_5 = y_5 + y_2, \quad z_6 = y_6 + y_2, \quad z_6 = y_3 - \tfrac{1}{2}y_2,$$

which would make $cA - B = 2y_2z_3 + 2z_4z_5 + z_6^2$; and $A = 2z_1y_2 + 2z_3z_4 + z_5^2 + z_6^2$, provided that $z_1 = y_1 + \frac{1}{2}y_2 - z_3 + \frac{1}{2}z_4 - z_5 - z_6$. For examples of the practical work, see Art. 19, Ex. 1, and Art. 22; and compare p. 42 (3_1), to (4_2).

Thus, by continuing the process of reduction, we find a succession of reduced terms of the type

$$\left.\begin{aligned} A &= 2x_1x_2 + 2x_3x_4 + \ldots, \\ cA - B &= 2x_2x_3 + 2x_4x_5 + \ldots, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where the notation for the variables has been changed, there being now no risk of confusion. The types of invariant-factors will depend on how the sequences terminate in equations (6); several cases must be examined :—

* This determinant is assumed to be different from zero, when we apply method (iv) to the reduction of $(cA - B)$ in the previous work.

First type.

$$A = 2x_1x_2 + 2x_3x_4 + \ldots + 2x_{2m-1}x_{2m} + kx^2_{2m+1} + f(x_{2m+2}, \ldots),$$
$$cA - B = 2x_2x_3 + 2x_4x_5 + \ldots + 2x_{2m}x_{2m+1} \qquad + g(x_{2m+2}, \ldots);$$

which corresponds to an invariant-factor $(\lambda - c)^{2m+1}$. The form of the determinant $|\lambda A - B|$ is illustrated by the case with $m = 2$, $k = 1$,

$$\begin{vmatrix} 0, & \lambda - c, & 0, & 0, & 0 \\ \lambda - c, & 0, & 1, & 0, & 0 \\ 0, & 1, & 0, & \lambda - c, & 0 \\ 0, & 0, & \lambda - c, & 0, & 1 \\ 0, & 0, & 0, & 1, & \lambda - c \end{vmatrix}$$

This determinant is equal to $(\lambda - c)^5$ and the minor of the first element is equal to 1, giving an invariant-factor $(\lambda - c)^5$.

Second type.

$$A = 2x_1x_2 + 2x_3x_4 + \ldots + 2x_{2m-1}x_{2m} + f(x_{2m+1}, \ldots),$$
$$cA - B = 2x_2x_3 + 2x_4x_5 + \ldots + kx^2_{2m} \qquad + g(x_{2m+1}, \ldots);$$

which corresponds to an invariant-factor $(\lambda - c)^{2m}$. Here the form of the determinant is (with $m = 2$, $k = 1$)

$$\begin{vmatrix} 0, & \lambda - c, & 0, & 0 \\ \lambda - c, & 0, & 1, & 0 \\ 0, & 1, & 0, & \lambda - c \\ 0, & 0, & \lambda - c, & 1 \end{vmatrix}$$

This is equal to $(\lambda - c)^4$, and the minor of the first zero is equal to -1, giving the invariant-factor $(\lambda - c)^4$.

Third type.

$$A = 2x_1x_2 + 2x_3x_4 + \ldots + 2x_{2m-1}x_{2m} \qquad + f(x_{2m+1}, \ldots),$$
$$cA - B = 2x_2x_3 + 2x_4x_5 + \ldots + 2x_{2m-2}x_{2m-1} + g(x_{2m+1}, \ldots):$$

which corresponds to two invariant-factors $(\lambda - c)^m$, $(\lambda - c)^m$. Illustrating again by taking $m = 2$, we have the determinant

$$\begin{vmatrix} 0, & \lambda - c, & 0, & 0 \\ \lambda - c, & 0, & 1, & 0 \\ 0, & 1, & 0, & \lambda - c \\ 0, & 0, & \lambda - c, & 0 \end{vmatrix}$$

which is equal to $(\lambda - c)^4$; while the minor of the last zero in the first row is $(\lambda - c)^2$ and one second minor is 1.

This type is obtained by the addition of two groups of the first or second types; and we can separate the parts in the reduced forms which correspond to the two invariant-factors by writing

$$\sqrt{2}y_k = x_k + x_{2m+1-k}, \quad \sqrt{2}y_{m+k} = x_k - x_{2m+1-k}, \qquad (k = 1, 2, \ldots, m).$$

Then if m is even the reduced parts are given by the table

A	$cA - B$
$(2y_1y_2 + \ldots + 2y_{m-1}y_m)$ $+ (2y_{m+1}y_{m+2} + \ldots + 2y_{2m-1}y_{2m})$	$(2y_2y_3 + \ldots + 2y_{m-2}y_{m-1} + y_m^2)$ $+ (2y_{m+2}y_{m+3} + \ldots + 2y_{2m-2}y_{2m-1} - y^2_{2m})$

When m is odd, the table is

A	$cA - B$
$(2y_1y_2 + \ldots + 2y_{m-2}y_{m-1} + y_m^2)$ $+ (2y_{m+1}y_{m+2} + \ldots + 2y_{2m-2}y_{2m-1} - y^2_{2m})$	$(2y_2y_3 + \ldots + 2y_{m-1}y_m)$ $+ (2y_{m+2}y_{m+3} + \ldots + 2y_{2m-1}y_{2m})$

The reader will find it instructive to verify these tables, say for $m = 2, 3$.

Fourth type.

$$A = 2x_1x_2 + 2x_3x_4 + \ldots + 2x_{2m-1}x_{2m} + f(x_{2m+2}, \ldots),$$
$$B = 2x_2x_3 + 2x_4x_5 + \ldots + 2x_{2m}x_{2m+1} + g(x_{2m+2}, \ldots);$$

this is a so-called *singular case** in which $|\lambda A - B|$ is identically zero for all values of λ. With $m = 2$, the determinant is

$$\begin{vmatrix} 0, & \lambda, & 0, & 0, & 0 \\ \lambda, & 0, & -1, & 0, & 0 \\ 0, & -1, & 0, & \lambda, & 0 \\ 0, & 0, & \lambda, & 0, & -1 \\ 0, & 0, & 0, & -1, & 0 \end{vmatrix}$$

In the first three types the reduced groups are completely determined by the corresponding invariant-factor, so that a knowledge of the invariant-factor is sufficient to enable us to write down the proper groups of terms in A, B; for there corresponds one and only one invariant-factor to each type of reduced forms. Consequently,

* It will probably be noticed that we have written B in place of $(cA - B)$. The reason is that as $|B|$ is zero, the introduction of $(cA - B)$ is superfluous.

when all the invariant-factors of $|\lambda A - B|$ are known, we can write down standard types to which A and B can be reduced by linear substitutions.

Example. The invariant-factors $(\lambda-1)^3$, $(\lambda-1)$, $(\lambda+1)^2$, give at once the reduced types

$$A = 2x_1x_2 + k_3x_3^2 + k_4x_4^2 + 2x_5x_6,$$
$$B = 2x_1x_2 + k_3x_3^2 + 2x_2x_3 + k_4x_4^2 - 2x_5x_6 + k_5x_5^2.$$

The fact that the reduced types depend *only* on the invariant-factors shows that the invariant-factors form a complete set of invariants of the two quadratic forms, except when the family is *singular*, so that the determinant vanishes identically for all values of λ; in the singular case additional invariants are needed, which we shall determine in the following article.

If the determinant $|B|$ is zero, no special remark is called for with reference to the terms which correspond to an invariant-factor λ^m; they are found by putting $c=0$. For example, the invariant-factor λ^3 would give the reduced parts

$$\text{in } A,\ 2x_1x_2 + kx_3^2; \quad \text{in } B,\ 2x_2x_3.$$

But if $|A|$ is zero there will be infinite roots of $|\lambda A - B| = 0$, and these must be considered specially; the simplest method is to take $|A - \mu B|$ and work out the groups corresponding to the invariant-factors of the form μ^m. The types of reduced parts may be deduced from the foregoing by taking in B the part found above for A, and in A the part found for $(cA - B)$. Thus an invariant-factor μ^3 would give reduced parts

$$\text{in } A,\ 2x_2x_3; \quad \text{in } B,\ 2x_1x_2 + kx_3^2,$$

and so on for other cases.

21. Determination of the invariants in the singular case.

If both the determinants $|A|$, $|B|$, are zero, it may happen that the determinant of the family $|\lambda A - B|$ is identically zero for all values of λ. In this case, we shall have one or more groups of reduced terms of the type

$$\left.\begin{array}{ll} & 2x_1x_2 + 2x_3x_4 + \dots + 2x_{2m-1}x_{2m} \text{ in } A \\ \text{and} & 2x_2x_3 + 2x_4x_5 + \dots + 2x_{2m}x_{2m+1} \text{ in } B \end{array}\right\} \dots\dots\dots\dots\text{(1)},$$

as in the fourth type of the last article. But we have at present no means of finding either the number of such groups or the number of variables in each group; and we shall now give rules for this purpose*.

* The recognition of the existence of singular cases is due to Kronecker, who also determined the suitable invariants; references to his papers will be given in Art. 38.

Since the determinant $|\lambda A - B|$ is zero, there will be (according to Arts. 4, 5) one or more linear relations amongst the n first derivates of $(\lambda A - B)$ with respect to the n variables. The coefficients in these relations will* involve λ; let us suppose them chosen so that the degree in λ of all the coefficients is the least possible. This is to exclude the possible multiplication of the relations by factors containing λ.

If now any linear substitution is applied to A and B, the first derivates of $(\lambda A - B)$ are also subjected to a linear substitution, *whose coefficients are independent of* λ; *thus the degrees of the relations mentioned in the last paragraph are invariants under the linear substitution.* We have now found suitable invariants; the only point remaining unsettled is the connection between these numbers and the number of terms in the reduced groups. Consider then the relation for the reduced groups (1), which contain $2m+1$ variables. We have

$$
\begin{aligned}
X_1 &= \tfrac{1}{2}\frac{\partial}{\partial x_1}(\lambda A - B) = \lambda x_2,\\
X_2 &= \tfrac{1}{2}\frac{\partial}{\partial x_2}(\lambda A - B) = \lambda x_1 - x_3,\\
X_3 &= \tfrac{1}{2}\frac{\partial}{\partial x_3}(\lambda A - B) = \lambda x_4 - x_2,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
X_{2m} &= \tfrac{1}{2}\frac{\partial}{\partial x_{2m}}(\lambda A - B) = \lambda x_{2m-1} - x_{2m+1},\\
X_{2m+1} &= \tfrac{1}{2}\frac{\partial}{\partial x_{2m+1}}(\lambda A - B) = \qquad\quad - x_{2m},
\end{aligned}
$$

so that the relation is

$$X_1 + \lambda X_3 + \lambda^2 X_5 + \dots + \lambda^m X_{2m+1} = 0,$$

and the *degree* of the relation is m.

Thus, if we find k independent relations amongst the first derivates, of least degrees $m_1, m_2, \dots, m_k$, there will be k reduced groups of the type (1), and the numbers of variables in these groups are

$$2m_1 + 1,\ 2m_2 + 1,\ \dots,\ 2m_k + 1.$$

These are the numbers which in the singular case correspond to the indices of some of the invariant-factors.

If the sum of these numbers $(= k + 2\Sigma m)$ is equal to n, no further invariants will be needed because the reduced forms will then contain n

* Except in the case, which we may suppose excluded, when A, B are functions of the same $(n-1)$ independent variables.

linearly independent variables. But it will usually happen that n is greater than $k+2\Sigma m$, in which case we must obtain other invariants in order to completely specify the reduced forms. To find these invariants, we observe that not all the kth minors can vanish identically, otherwise there would be $(k+1)$ relations amongst the first derivates of $(\lambda A - B)$, instead of only k, as we assumed above. We find then the H.C.F. of the non-zero minors of the kth order and factorize this H.C.F. just as we factorize the determinant in the non-singular cases. Take $(\lambda - c)$ as a typical factor, with index l_k: and let l_{k+1} be the index of $(\lambda - c)$ in the H.C.F. of the $(k+1)$th minors, and so on; then the indices of the invariant-factors with the base $(\lambda - c)$ are the differences $l_k - l_{k+1},\ l_{k+1} - l_{k+2}, \ldots$.

To illustrate, take the determinant

$$\begin{vmatrix} 0, & \lambda, & 0, & 0, & 0, & 0, & 0, & 0 \\ \lambda, & 0, & -1, & 0, & 0, & 0, & 0, & 0 \\ 0, & -1, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & \lambda, & 0, & 0, & 0 \\ 0, & 0, & 0, & \lambda, & 0, & -1, & 0, & 0 \\ 0, & 0, & 0, & 0, & -1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & \lambda - c \\ 0, & 0, & 0, & 0, & 0, & 0, & \lambda - c, & -1 \end{vmatrix}$$

for which we find $k=2,\ m_1=1,\ m_2=1$; but the second minor found by striking out the first and fourth rows and columns is $(\lambda - c)^2$; and the third minor found from the first, fourth and seventh rows and columns is 1. These facts indicate two groups of the type (1), each containing three variables, and one invariant-factor $(\lambda - c)^2$.

It is an obvious extension of the symbol employed for invariant-factors to denote this case by the characteristic [{33} 2]; where the figures in crooked brackets { } give the numbers of variables in the separate groups of type (1) and the other figures represent indices of invariant-factors as usual.

Perhaps it might be thought better to use the numbers m rather than the numbers $(2m+1)$; for m is the number given directly by our rule, and $(2m+1)$ is at once deducible from m. The advantage of using $(2m+1)$ lies in the fact that, by adopting this as the standard, every number in the characteristic symbol represents the number of variables in a certain group of terms. Thus in all cases *by adding all the numbers in the characteristic, we obtain the number of independent variables in the family of forms.*

22. Example of reducing a singular case.

In view of the examples already given in Arts. 16, 19 it is probably unnecessary to give more than one here: this one is Darboux's example (p. 375 of his paper quoted in Art. 38 below) of a singular family of quadratic forms, for which $|\lambda A - B| \equiv 0$.

(i) Consider first

$$A = x_1x_2 + x_3x_4 + c(x_2x_3 + x_4x_5), \quad B = x_2x_3 + x_4x_5.$$

The variable x_1 occurs in A and not in B; but x_1^2 is not present in A, so that we must write

$$A = x_1'x_2 + (x_3x_4 + cx_4x_5), \quad B = x_2x_3 + x_4x_5 \quad\text{.................(1)},$$

where $x_1' = x_1 + cx_3$, or $x_1 = x_1' - cx_3$.

It will be seen that equations (1) correspond to equations (5_2) of Art. 20, with $h_1 = x_3x_4 + cx_4x_5$, $f_2 = x_4x_5$. To reduce h_1 and f_2, we have to write

$$x_3' = x_3 + cx_5, \text{ or } x_3 = x_3' - cx_5.$$

The complete forms then become

$$A = x_1'x_2 + x_3'x_4, \quad B = x_2(x_3' - cx_5) + x_4x_5.$$

If we follow the method explained in the small type on p. 52 of Art. 20, we shall write

$$x_4' = x_4 - cx_2, \text{ and so } B = x_2x_3' + x_4'x_5;$$

and then $\quad A = x_1'x_2 + x_3'(x_4' + cx_2) = (x_1' + cx_3')x_2 + x_3'x_4'.$

Thus the reduced forms are

$$A = \xi_1\xi_2 + \xi_3\xi_4, \quad B = \xi_2\xi_3 + \xi_4\xi_5 \quad\text{........................(2)},$$

where $\quad \xi_1 = x_1' + cx_3', \quad \xi_2 = x_2, \quad \xi_3 = x_3', \quad \xi_4 = x_4', \quad \xi_5 = x_5.$

The reducing substitution is most conveniently written:—

$$\left.\begin{array}{ll} x_1 = \xi_1 - 2c\xi_3 + c^2\xi_5, & \xi_2 = x_2, \\ x_3 = \xi_3 - c\xi_5, & \xi_4 = x_4 - cx_2, \\ x_5 = \xi_5. & \end{array}\right\}\text{......................(3)}.$$

(ii) Consider next

$$A = x_1x_2 + x_3x_4 + x_5x_6 + c(x_2x_3 + x_4x_5 + x_6x_7),$$
$$B = x_2x_3 + x_4x_5 + x_6x_7.$$

Following the same rule as in (i) we find at once

$$A = x_1'x_2 + A', \quad B = x_2x_3 + B' \quad\text{.......................(4)},$$

where $x_1' = x_1 + cx_3$ (or $x_1 = x_1' - cx_3$), and A', B' are of the same type as the forms reduced in (i). Thus we shall have the reduced forms

$$A = x_1'x_2 + \xi_3\xi_4 + \xi_5\xi_6, \quad B = x_2x_3 + \xi_4\xi_5 + \xi_6\xi_7,$$

where $$x_3 = \xi_3 - 2c\xi_5 + c^2\xi_7,$$

the other variables ξ being given by equations similar to (3), with changed suffixes. When this value of x_3 is inserted in B, we find

$$B = x_2\xi_3 + \xi_4'\xi_5 + \xi_6'\xi_7, \text{ if } \xi_4' = \xi_4 - 2cx_2, \quad \xi_6' = \xi_6 + c^2x_2.$$

Then $A=x_1'x_2+\xi_3(\xi_4'+2cx_2)+\xi_5(\xi_6'-c^2x_2)=\xi_1x_2+\xi_3\xi_4'+\xi_5\xi_6'$,

where $\xi_1=x_1'+2c\xi_3-c^2\xi_5$, or $x_1=x_1'-cx_3=\xi_1-3c\xi_3+3c^2\xi_5-c^3\xi_7$.

We may write then finally

$$A=X_1X_2+X_3X_4+X_5X_6,\quad B=X_2X_3+X_4X_5+X_6X_7\ \ldots\ldots\ldots(5),$$

where

$$X_1=\xi_1,\quad X_2=x_2,\quad X_3=\xi_3,\quad X_4=\xi_4',\quad X_5=\xi_5,\quad X_6=\xi_6',\quad X_7=\xi_7.$$

The reducing substitution is :—

$$\left.\begin{aligned} x_1&=X_1-3cX_3+3c^2X_5-c^3X_7, & X_2&=x_2,\\ x_3&=X_3-2cX_5+c^2X_7, & X_4&=-2cx_2+x_4,\\ x_5&=X_5-cX_7, & X_6&=c^2x_2-cx_4+x_6,\\ x_7&=X_7. \end{aligned}\right\}\ \ldots\ldots(6).$$

(iii) We can now generalize the work in (i) and (ii), to similar forms in $(2n-1)$ variables. With a slight change of notation, we take

$$A=x_1y_1+x_2y_2+\ldots+x_{n-1}y_{n-1}+c(x_2y_1+x_3y_2+\ldots+x_ny_{n-1}),$$
$$B=x_2y_1+x_3y_2+\ldots+x_ny_{n-1}.$$

Then the reduced forms are

$$A=\xi_1\eta_1+\xi_2\eta_2+\ldots+\xi_{n-1}\eta_{n-1},\quad B=\xi_2\eta_1+\xi_3\eta_2+\ldots+\xi_n\eta_{n-1}\ldots\ldots(7),$$

and the necessary substitutions are given, symbolically, by

$$\left.\begin{gathered} x_r=\xi^r(1-c\xi)^{n-r},\qquad \xi^r=\xi_r,\qquad (r=1,2,\ldots,n),\\ \eta_1+t\eta_2+t^2\eta_3+\ldots+t^{n-2}\eta_{n-1}=(1-ct)^{n-2}y_1+(1-ct)^{n-3}y_2+\ldots+y_{n-1}. \end{gathered}\right\}(8).$$

The reader should find no difficulty in establishing the truth of (7), (8) by induction from (2), (3) and (5), (6).

It is somewhat curious that the substitutions inverse to (8) are given by

$$\left.\begin{gathered} \xi_r=x^r(1+cx)^{n-r},\qquad x^r=x_r,\qquad (r=1,2,\ldots,n),\\ y_1+ty_2+t^2y_3+\ldots+t^{n-2}y_{n-1}=(1+ct)^{n-2}\eta_1+(1+ct)^{n-3}\eta_2+\ldots+\eta_{n-1}. \end{gathered}\right\}(9).$$

The similarity of the substitutions giving ξ_r in terms of x_r and x_r in terms of ξ_r is substantially the result set in the Mathematical Tripos, 1905, at the beginning of the second problem paper.

23. Characteristics of forms containing more than four variables.

The reader should now have no difficulty in constructing reduced forms having any prescribed characteristic; but it may be useful to give some information as to the possible characteristics.

Of families in 5 variables, there are 10 algebraically distinct types, whose characteristics are given in the following list; and each type can be divided further by equating certain coefficients in the reduced forms. In this way, the 10 types give rise to 31 cases, whose characteristics

are obtained by adding round brackets wherever possible. The types are represented by :—

[11111]	7 cases	[41]	2 cases
[2111]	7 ,,	[5]	1 ,,
[221]	4 ,,	[{3} 11]	2 ,,
[311]	4 ,,	[{3} 2]	1 ,,
[32]	2 ,,	{5}	1 ,,

For 6 variables there are 16 distinct types, as follows :—

[111111]	[3111]	[411]	[{3} 111]	[{5} 1]
[21111]	[321]	[42]	[{3} 21]	
[2211]	[33]	[51]	[{3} 3]	
[222]		[6]	[{3} {3}]	

These 16 types correspond to 66 cases, 8 of which are singular families.

For details as to the reduced forms, the reader may consult the references in Arts. 35, 36 below; but it should be noted that some of the cases are omitted there for geometrical reasons.

In conclusion we give a table* shewing the number of types and cases possible for each value of n from 1 to 11 :—

	n	1	2	3	4	5	6	7	8	9	10	11
Non-singular	Types	1	2	3	5	7	11	15	22	30	42	56
	Cases	1	3	6	14	27	58	111	223	424	817	1527
Singular	Types	–	–	1	1	3	5	9	14	24	36	56
	Cases	–	–	1	1	4	8	19	38	84	165	338

* This table was given for the non-singular cases by Sylvester (*Phil. Mag.* series 4, vol. VII, 1854, p. 334; *Coll. Papers*, vol. II, pp. 30—33); these numbers have been re-calculated and also compared with a MS. table of Cayley's. But no list has been accessible for checking the singular cases.

The number of non-singular types (c_n) is the coefficient of x^n in the product

$$X_1 = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}\ldots = 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots;$$

and the number of non-singular cases is the coefficient of x^n in the product

$$X_2 = \prod_{r=1}^{\infty}(1-x^r)^{-c_r} = (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-5}\ldots$$

The numbers for the singular families are found similarly from the products X_1Y and X_2Y, where

$$Y = -1 + \prod_{r=1}^{\infty}(1-x^{2r+1})^{-1} = x^3 + x^5 + x^6 + x^7 + x^8 + 2(x^9 + x^{10} + x^{11}) + \ldots$$

CHAPTER IV.

THEOREMS ON THE INVARIANT-FACTORS OF A FAMILY OF QUADRATIC FORMS.

24. Having established the possibility of reducing a family of forms to groups of the types found in Art. 20, we must now calculate directly the invariant-factors of the complete family, and prove that they are the same as the invariant-factors of the separate groups. It is clear from what was proved in Art. 11 (p. 29) that we need only examine the effect of combining groups whose invariant-factors have the *same* base; for we have seen that when the bases are different, the invariant-factors of the whole family are made up of those belonging to the separate parts.

Consider then the family

$$\lambda A - B = \sum_{k=1}^{r} (\lambda A_k - B_k) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where A_k and B_k denote groups of terms of the first or second types* in Art. 20, so that we can write

$$D_k = |\lambda A_k - B_k| = \pm (\lambda - c)^{a_k} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where a_k denotes the number of variables in the groups A_k, B_k. It must be remembered that the minor of the first element in D_k is ± 1, so that $(\lambda - c)^{a_k}$ is the invariant-factor of $|\lambda A_k - B_k|$. Thus

$$\Delta = |\lambda A - B| = D_1 D_2 \ldots D_r \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and accordingly the index of $(\lambda - c)$ in Δ is

$$l_0 = a_1 + a_2 + a_3 + \ldots + a_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

* The third type has been seen (p. 54) to be not substantially different from the first and second. If groups of terms of the fourth type occur, the following argument has to be modified somewhat, in consequence of the determinant $|\lambda A - B|$ being identically zero. But the reader should have little difficulty in seeing that the final result is the same. It may be useful to remark that if A_k and B_k belong to the fourth type, the determinant D_k is identically zero, but the minor of the first element in D_k is still ± 1.

Now Δ can be written in the symbolical form

$$\Delta = \begin{vmatrix} D_1, & \omega, & \ldots, & \omega \\ \omega, & D_2, & \ldots, & \omega \\ \multicolumn{4}{c}{\dotfill} \\ \omega, & \omega, & \ldots, & D_r \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where each ω denotes a rectangular *block* of zeros. From (5) it is plain that the first minor of any element in a block ω will be zero. Further, the first minor in Δ of any element in D_1 is equal to the product of the corresponding first minor in D_1 by $D_2, D_3, \ldots, D_r$. Thus the lowest index of $(\lambda - c)$ which divides *all* first minors in Δ is

$$l_1 = a_2 + a_3 + a_4 + \ldots + a_r \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

provided that the groups are arranged so that

$$a_1 \geqq a_2 \geqq a_3 \geqq \ldots \geqq a_r \geqq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7);$$

and l_1 is exactly the index of $(\lambda - c)$ in the minor of the first element of Δ.

Similarly, by striking out the first element of D_1 and the first element of D_2, we get a second minor of Δ which has the index

$$l_2 = a_3 + a_4 + \ldots + a_r \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and no other second minor has a lower index. And, continuing the argument, we see that the sequence of indices $l_0, l_1, l_2, \ldots$ ends with

$$l_{r-1} = a_r, \quad l_r = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

From (6), (8), (9), it is plain that the invariant-factors of Δ have indices

$$e_1 = l_0 - l_1 = a_1, \quad e_2 = l_1 - l_2 = a_2, \ \ldots, \ e_r = l_{r-1} = a_r \quad \ldots\ldots (10).$$

That is, **the invariant-factors of the complete family are the same as the invariant-factors of the separate groups.** This completes the proof of the theorem stated on p. 29.

It is also useful to note that **an invariant-factor of any order is divisible by the following invariant-factor to the same base***; this is obvious from (7) and (10). And, if we can prove that $e_1 = 1$, it follows that $e_2, e_3, \ldots, e_r$ must also be 1 (compare Arts. 26, 27).

The reader is advised to examine some special cases, in order to follow the general account more readily; for example

$$\lambda A - B = (2\lambda x_1 x_2 + \lambda x_3^2 + 2x_2 x_3) + (2\lambda x_4 x_5 + x_5^2) + \lambda x_6^2,$$

* This theorem is of course a theorem on minors of determinants and *can* be proved without reference to reduced forms. A proof of this character, due to Frobenius, will be found in Muth's *Elementartheiler*, pp. 7—11; but it can hardly be regarded as easy.

where $$D_1 = \begin{vmatrix} 0, & \lambda, & 0 \\ \lambda, & 0, & 1 \\ 0, & 1, & \lambda \end{vmatrix}, \quad D_2 = \begin{vmatrix} 0, & \lambda \\ \lambda, & 1 \end{vmatrix}, \quad D_3 = \lambda,$$

and the complete determinant Δ should be written out.

25. Regular Minors.

A minor of the sth order is called regular with respect to a particular base $(\lambda - c)$, if it contains $(\lambda - c)$ to a power not higher than l_s (following the notation of Art. 11). Thus in the last article, the minor found by striking out the first element of D_1 is regular; so also is the second minor found by striking out the first elements of D_1 and D_2; and so is the third minor found from the first elements of D_1, D_2, D_3; and so on. And these are the only regular minors of their respective orders.

From these results it may be guessed that *a regular first minor contains at least one regular second minor*: and that *a regular second minor contains at least one regular third minor, and so on**. The inference is correct; but an additional proof is required, because it is not obvious that the property is invariant under a linear substitution.

Another inference might also be made; that regular *symmetrical* minors can be found. This is usually, but not always, possible as may be seen from the determinant

$$\begin{vmatrix} 0, & \lambda, & 0, & 0 \\ \lambda, & 0, & 1, & 0 \\ 0, & 1, & 0, & \lambda \\ 0, & 0, & \lambda, & 0 \end{vmatrix}$$

in which the regular first minors are found by striking out the first row and last column; or else the last row and first column. It may be noted that when the regular minors are all asymmetrical, two or more of the invariant-factors must be *equal*. In the determinant above there are two invariant-factors equal to λ^2.

Without going into the general proof, we shall indicate how to verify the property that a regular minor contains regular sub-minors, in the example of the last article. By Art. 10, equation (4), we obtain a symmetrical first minor of the transformed determinant by bordering with $m_1, m_2, m_3, m_4, m_5, m_6, \ldots$ if

$$m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + m_5x_5 + m_6x_6 + \ldots$$

* This theorem is again one of a purely determinantal character; the proof quoted in the last note establishes this result also.

is one of the new variables. It will be found that if δ denotes the determinant of the rest of the family (*not divisible by* λ), this minor contains the term $\lambda^3\delta m_1^2$ and that all the other terms contain λ^4 as a factor, so that the minor is regular if m_1 is not zero.

In like manner, any second minor, *derived from this first minor*, can be obtained by bordering with m's and n's. It is easily seen that this minor contains a term $\lambda\delta(m_1n_4 - m_4n_1)^2$ and that all other terms contain λ^2 as a factor: the minor is therefore regular if $(m_1n_4 - m_4n_1)$ is different from zero. This cannot vanish for *all* the new variables other than Σmx; for, if so, we should have* $n_4 = m_4n_1/m_1$ and therefore

$$n_1x_1 + n_4x_4 = (n_1/m_1)(m_1x_1 + m_4x_4).$$

That is to say, x_1 and x_4 would only appear in the new set of variables through the combination $(m_1x_1 + m_4x_4)$; and so the new set of variables could not be linearly independent.

Similarly the minor found by adding a row and column of p's will be regular if the determinant

$$\begin{vmatrix} m_1, & n_1, & p_1 \\ m_4, & n_4, & p_4 \\ m_6, & n_6, & p_6 \end{vmatrix}$$

is not zero; and it can be proved in the same kind of way that this determinant is not zero when the new variables are linearly independent.

26. If one form A belonging to a family of real quadratic forms is definite, all the roots of the equation $|\lambda A - B| = 0$ are real; and the invariant-factors of the determinant are all linear.

Consider first the theorem that the roots are real; amongst the numerous proofs that have been published, perhaps the following (due to Kronecker) is the simplest.

Suppose if possible that $\lambda = a + ib$ is a complex root of $|\lambda A - B| = 0$; then, according to Arts. 5, 8 the form

$$(a + ib)A - B$$

is expressible as the sum of, at most, $(n-1)$ squares of complex variables, taking n as usual to be the number of independent variables contained in the family $(\lambda A - B)$.

Thus

$$(a + ib)A - B = \sum_{r=1}^{n'} a_r(y_r + iz_r)^2 \qquad (n' \leqq n - 1),$$

* Note that m_1 is not zero because the first minor is *regular*.

where the y's and z's are *real* linear combinations of the original variables. Since we are working with complex numbers, we may suppose the square-roots of the coefficients a_r taken into the brackets; this change will simply change* y_r and z_r; and then

$$(a + ib) A - B = \sum_{r=1}^{n'} (y_r + iz_r)^2 \qquad (n' \leqq n - 1).$$

Now the coefficients of A, B are real; and consequently, on equating real and imaginary parts, we must have

$$bA = 2 \sum_{r=1}^{n'} y_r z_r.$$

But the product $y_r z_r$ can be made zero by taking either of its factors to be zero; and consequently, by satisfying n' linear relations amongst the original variables (say $y_1 = 0$, $y_2 = 0$, ..., $y_{n'} = 0$), we can make bA zero. Now n' is less than n, and consequently these n' relations give *real* values for the original variables, which *are not all zero*. But (Art. 9) the form A, being definite, *cannot* vanish for such values of the variables; hence $b = 0$, and therefore the roots of the equation $|\lambda A - B| = 0$ must be real.

The following proof that the invariant-factors are linear is also due to Kronecker. Let $\lambda = c$ be a root of the determinantal equation; then we can choose the variables $x_1, x_2, \ldots, x_n$, so that $(cA - B)$ contains only $x_2, x_3 \ldots, x_n$ Thus A can be brought, by using method (i) or (ii) of Art. 7, to one of the two forms

$$A = y_1^2/a_{11} + f(x_2, x_3, \ldots, x_n),$$

or

$$A = 2y_1 y_2 + g(x_3, x_4, \ldots, x_n),$$

where y_1 is a real linear function of $x_1, x_2, \ldots, x_n$, and y_2 of $x_2, x_3, \ldots, x_n$. In the second case it is easy to see that $|\lambda A - B|$ contains the factor $(\lambda - c)^2$; so, if the roots of $|\lambda A - B| = 0$ are all different, we have the first case only, without making any restriction in regard to the nature of A. Again, in the second case A vanishes for *all* values of y_1, provided that $y_2 = 0$, $x_3 = 0$, ..., $x_n = 0$. This is impossible when A is a definite form; so that whether there are equal roots of $|\lambda A - B| = 0$ or not, we can always bring $(\lambda A - B)$ to the form

$$(\lambda - c)\, y_1^2/a_{11} + \lambda A' - B',$$

where A', B' contain only $(n - 1)$ variables.

* If $a_r = (\alpha_r + i\beta_r)^2$ we see that

$$(\alpha_r + i\beta_r)(y_r + iz_r) = \alpha_r y_r - \beta_r z_r + i(\alpha_r z_r + \beta_r y_r)$$

so that we get other real linear combinations of the original variables.

Proceeding in this way we finally reduce A and B to sums of the same n squares, whether there are equal roots of $|\lambda A - B| = 0$ or not. On writing out the reduced form of the determinant, it is plain that all the invariant-factors are linear.

Example. $\begin{vmatrix} \lambda - c, & 0, & 0, & 0 \\ 0, & \lambda - c, & 0, & 0 \\ 0, & 0, & \lambda - c, & 0 \\ 0, & 0, & 0, & \lambda - c' \end{vmatrix}$ has linear invariant-factors.

The fact that all the roots of the equation $|\lambda A - B| = 0$ are real seems to have been noticed first by Lagrange (in 1773) for the special case of three variables and $A = x_1^2 + x_2^2 + x_3^2$; the case of n variables and $A = \Sigma x_r^2$ was given by Cauchy (in 1829). The earliest treatment of the general case, with any definite form for A, seems to have been given by Sylvester (*Phil. Mag.* Series 4, vol. VI, 1853, p. 214; *Coll. Papers*, vol. I, p. 634); the theorem was rediscovered by Weierstrass (in 1858). Routh (*Rigid Dynamics*, vol. II, § 58) attributes the theorem to Lord Kelvin, but the earliest reference to it by Lord Kelvin seems to occur in the first edition of Thomson and Tait's *Natural Philosophy* (1867), § 343, where the authors allude to the theorem as having escaped the notice of modern analysts; so that apparently Sylvester and Weierstrass have the priority.

The fact that all the invariant-factors are linear is due to Weierstrass; the result was published in 1858 and is contained in § 4 of his first paper quoted in Art. 38. This theorem has been rediscovered by a number of authors, Somoff, Villarceau, Jordan and Routh being the principal names. For more details see Art. 37.

27. Semi-definite forms.

It happens in some cases (*e.g.* Art. 31) that the form A, although not definite, is *semi-definite*. In such cases, the arguments of the last article would fail, as they stand; because a semi-definite form may vanish for some real non-zero values of the variables.

If, as before, the family $(\lambda A - B)$ contains n independent variables, the form A can be expressed in terms of fewer than n variables. Let new variables be chosen so that $x_1, x_2, \ldots, x_p$ are the only ones present in A (its rank being p); then $x_{p+1}, x_{p+2}, \ldots, x_n$ are present in B but not in A.

If x_n^2 is present in B, a new variable y_n can be introduced (Art. 7 (i)) so that all the terms in x_n are absorbed into y_n^2; this change does not alter A. Similarly, if x_n^2 is absent, but a product

such as $x_n x_{n-1}$ is present, all the terms in x_n and in x_{n-1} can be absorbed into a product $y_n y_{n-1}$ (Art. 7 (ii)), which can then be put in the form $z_n^2 - z_{n-1}^2$, as in Art. 8; and again A has not been changed.

But if x_n only appears in B through products involving $x_1, x_2, \ldots, x_p$, the case is different; for example, suppose that x_n appears only in the form of the product $x_n(x_1 + x_2)$. We write then $y_1 = x_1 + x_2$, and by introducing a suitable new variable y_n, we can absorb into the product $y_1 y_n$ all terms in B which contain x_1 or x_n. The form A is then modified; but A will still contain only p variables, $y_1, x_2, x_3, \ldots, x_p$.

Proceeding in this manner, B will be reduced so as to contain, first, a certain number $(n-p-q)$ of squares of variables not present in A; and secondly a certain number (q) of products of two variables, one of which is present in A and the other is not; and finally a quadratic expression containing only variables present in A.

Thus, the forms can be written, the suffixes being possibly rearranged:—

$$A = f(y_1, y_2, \ldots, y_p),$$
$$B = g(y_1, y_2, \ldots, y_{p-q}) + 2\Sigma y_r y_{q+r} + \Sigma b_s y_s^2$$
$$(r = p-q+1,\ p-q+2, \ldots, p; \quad s = p+q+1,\ p+q+2, \ldots, n),$$

where the number q, of course, cannot exceed either p or $(n-p)$; and the new variables $y_1, y_2, \ldots, y_p$ are *real* linear combinations of $x_1, x_2, \ldots, x_p$ and $y_{p+1}, y_{p+2}, \ldots, y_n$ are *real* linear combinations of (possibly) all the x's. It is now to be observed that A is definite, regarded as a function of $y_1, y_2, \ldots, y_p$; and therefore method (iii) of Art. 7 is always applicable* to A. Thus we can write

$$A = f(y_1', y_2', y_3', \ldots, y_{p-1}', 0) + k y_p^2,$$

where $$(y_1' - y_1),\ (y_2' - y_2),\ \ldots,\ (y_{p-1}' - y_{p-1})$$

are simply multiples of y_p; and therefore the change produced in B can all be absorbed into the product $y_p y_{p+q}$, by modifying y_{p+q}. Continuing the process, we find

$$A = f_1(z_1, z_2, \ldots, z_{p-q}) + \Sigma a_r z_r^2,$$
$$B = g_1(z_1, z_2, \ldots, z_{p-q}) + 2\Sigma z_r z_{r+q} + \Sigma b_s y_s^2,$$

where, since all the transformations have *real* coefficients, f_1 is a definite form and all the coefficients a_r have the same sign as the form A. Thus, by inspection we see that the determinant $|\lambda A + \mu B|$ has the

* For, if method (iv) had to be used, A would take a form such as

$$2y_{p-1} y_p + f(z_1, z_2, \ldots, z_{p-2}, 0, 0)$$

which is not definite.

same invariant-factors as $|\lambda f_1 + \mu g_1|$ together with q invariant-factors of the type μ^2 and $(n-p-q)$ of the type μ. To make the result clearer, we give an example of the determinant, taking

$$A = 2z_1^2 + 2z_1z_2 + 5z_2^2 + z_3^2 + z_4^2,$$
$$B = 3z_1^2 + 6z_1z_2 + 9z_2^2 + 2z_3z_5 + 2z_4z_6 - z_7^2,$$

so that $$p = 4,\ q = 2,\ n = 7.$$

Then $|\lambda A + \mu B|$ is

$$\begin{vmatrix} 2\lambda+3\mu, & \lambda+3\mu, & 0, & 0, & 0, & 0, & 0 \\ \lambda+3\mu, & 5\lambda+9\mu, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & \lambda, & 0, & \mu, & 0, & 0 \\ 0, & 0, & 0, & \lambda, & 0, & \mu, & 0 \\ 0, & 0, & \mu, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & \mu, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & -\mu \end{vmatrix}$$

which has the invariant-factors μ^2, μ^2, μ, as well as the two $(\lambda+\mu)$, $(\lambda+2\mu)$, from the determinant

$$\begin{vmatrix} 2\lambda+3\mu, & \lambda+3\mu \\ \lambda+3\mu, & 5\lambda+9\mu \end{vmatrix}.$$

With regard to the invariant-factors of $|\lambda f_1 + \mu g_1|$, the arguments of Art. 26 can be at once applied to shew that they are all real and linear; so that we have now the theorem:—

If one form A of a family of forms $\lambda A + \mu B$ is semi-definite, all the invariant-factors of the determinant $|\lambda A + \mu B|$ are real and linear, with the possible exception that some may be of the type μ^2; the number of such invariant-factors cannot exceed the smaller of the numbers p, $(n-p)$.

Here p denotes the rank of A and n is the number of independent variables in the family of forms.

28. If a form A of the family (although not definite) has a known *signature* (p. 22), there is a theorem due to Klein which gives information with respect to the invariant-factors of the determinant $|\lambda A - B|$.

The methods of investigation of Arts. 26, 27 fail here and we simply appeal to the reduced forms obtained in Chapter III.

In the first place we recall the fact that if the coefficients of the given forms are real no complex numbers will be introduced in the reducing process except through the factors of the determinant; that is, if $\lambda = c$ is a real root of $|\lambda A - B| = 0$, the manipulation of A and

$(cA - B)$ will not introduce any complex coefficients in the transformation. Thus in the reduced forms the variables which correspond to real invariant-factors*, will be real linear combinations of the original variables.

Now consider the effect on the signature of A of the various types of reduced groups; we see from Art. 20 that the only type which contributes to the signature is that given by an invariant-factor $(\lambda - c)^{2m+1}$. For a product such as $2x_1x_2$ can be written

$$\tfrac{1}{2}(x_1 + x_2)^2 - \tfrac{1}{2}(x_1 - x_2)^2;$$

and since this contains one positive and one negative square, it adds 1 to and subtracts 1 from the signature; so that such a term cannot alter the signature.

But if the determinant $|A|$ is zero, there will be invariant-factors to base μ belonging to the determinant $|\lambda A + \mu B|$; and in this case, those of the type μ^{2m} are the only ones to affect the signature of A.

Thus, so far as the real invariant-factors are concerned, the signature cannot exceed the sum of the number of invariant-factors of the types $(\lambda - c)^{2m+1}$, μ^{2m}.

Consider now the complex invariant-factors. If $(\lambda - \alpha - i\beta)^3$ is one of these†, $(\lambda - \alpha + i\beta)^3$ is another, and the corresponding sets of variables are conjugate complexes; thus the two together give a contribution to A of the form

$$\begin{aligned} 2(y_1 + iz_1)(y_2 + iz_2) + (p + iq)(y_3 + iz_3)^2 \\ + 2(y_1 - iz_1)(y_2 - iz_2) + (p - iq)(y_3 - iz_3)^2 \\ = 4(y_1y_2 - z_1z_2) + 2p(y_3^2 - z_3^2) - 4qy_3z_3. \end{aligned}$$

But these terms add nothing to the signature, since we may write

$$p(y_3^2 - z_3^2) - 2qy_3z_3 = p(y_3 + qz_3/p)^2 - (p^2 + q^2)z_3^2/p,$$

when p is not zero; and in case p is zero, the term adds nothing to the signature, as before.

Since the signature is invariant under real linear substitutions, we have now Klein's theorem :—

If the determinant $|A|$ is not zero, the number of real invariant-factors with odd indices is not less than the signature of A.

This theorem was given in Klein's Inauguraldissertation (Bonn, 1868), reprinted in the *Mathematische Annalen*, Bd. 23, p. 539; see in particular p. 562. But Sylvester had previously proved that the

* That is, invariant-factors of the type $(\lambda - c)^e$, where c is real.

† To simplify the argument, we take only the index 3: but the method and results are perfectly general.

number of real roots of $|\lambda A - B|$ cannot be less than the signature of any form in the family (*Phil. Mag.* series 4, vol. VI, 1853, p. 216; *Coll. Papers*, vol. I, p. 635). Sylvester's method is most instructive, but it does not seem to be capable of giving as much precision as Klein's, which is explained above.

When $|A|$ is zero, Klein's theorem must be modified, and the new result is :—

The signature of A cannot exceed the total number of real invariant-factors belonging to the types $(\lambda - c)^{2m+1}$, μ^{2m}.

This theorem includes, in part, those of Arts. 26, 27; and of course a similar method of proof might have been used in those articles. But it seemed preferable to give an alternative process.

Klein's theorem has been extended by A. Loewy and the present author* to Hermite's forms containing complex variables.

* A. Loewy, Crelle's *Journal*, Bd. 122, 1900, p. 56; Bromwich, *Proc. Lond. Math. Soc.*, vol. XXXII, 1900, p. 349.

CHAPTER V.

APPLICATIONS AND REFERENCES.

29. Metrical classification of conics and quadrics (point-coordinates).

It may appear almost superfluous to discuss this problem here, as all the necessary details are given in the ordinary text-books on analytical geometry. But it is of some interest to shew the use that can be made of the results in the previous chapters; and it may be useful to describe a simple but general method of classification, which is not given in the ordinary books.

For brevity, we confine the work to quadrics since the necessary changes in the case of conics will be recognised without difficulty, and a list of results for conics will be found on p. 75.

If the equation to the quadric is

$$(a, b, c, f, g, h \between x, y, z)^2 + 2(ux + vy + wz) + d = 0 \quad\ldots\ldots(1),$$

usually the first step is to transform the terms of second degree, say

$$B = (a, b, c, f, g, h \between x, y, z)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

so as to become a sum of squares, the transformation being *orthogonal*; that is, the quadratic form

$$A = x^2 + y^2 + z^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

must remain of the same type when the new variables are introduced*. Thus the problem is reduced to the transformation of A, B to canonical

* It is assumed that the original coordinates are orthogonal; but if they are oblique we start from

$$A = x^2 + y^2 + z^2 + 2lyz + 2mzx + 2nxy,$$

l, m, n, being the cosines of the angles between the axes.

forms; and since A is a *definite* form, the determinant $|\lambda A - B|$ has only real linear invariant-factors (Art. 26). Hence a real set of variables can be found so that

$$A = x_1^2 + x_2^2 + x_3^2, \quad B = c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 \qquad \text{(4)}.$$

The equation to the quadric then becomes

$$c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 + 2(u_1 x_1 + u_2 x_2 + u_3 x_3) + d = 0,$$

and the subsequent reduction depends on the number of the coefficients c_1, c_2, c_3 which are zero*. It is easily seen, as in the text-books, that there are three fundamental reduced types :—

First $$c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2 + c_4 = 0 \qquad (5_1),$$

where one or two of c_1, c_2, c_3 may be zero.

Secondly $$c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2 = 0 \qquad (5_2),$$

where one or two of c_1, c_2, c_3 may be zero.

Thirdly $$c_1 y_1^2 + c_2 y_2^2 + 2k y_3 = 0 \qquad (5_3),$$

where c_3 is necessarily zero and one of the two c_1, c_2 may also be zero.

Thus in all cases we have the possible type

$$c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2 + 2k y_3 + c_4 = 0 \qquad (5_4),$$

in which either k or c_3 will be zero; we shall now shew how to determine the coefficients and the type *directly* from the given quadric, without having to determine the reducing substitution.

We have the identity

$$(a, b, c, f, g, h \between x, y, z)^2 - \lambda(x^2 + y^2 + z^2) + 2(ux + vy + wz) + d$$
$$= c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2 + 2k y_3 + c_4 - \lambda\left[(y_1 + p_1)^2 + (y_2 + p_2)^2 + (y_3 + p_3)^2\right].$$

* Thus if c_1, c_2, c_3 are all different from zero, we can write

$$y_r = x_r + u_r/c_r, \qquad (r = 1, 2, 3),$$

and the equation becomes $\quad \Sigma c_r y_r^2 + d - \Sigma u_r^2/c_r = 0.$

But if c_3 is zero, the type becomes

$$c_1 y_1^2 + c_2 y_2^2 + 2u_3 x_3 + d - (u_1^2/c_1 + u_2^2/c_2) = 0.$$

In some of the books, it is advised that the terms of the first degree should be absorbed, as far as possible, before transforming those of the second degree; if so, we virtually apply Art. 7 (iii) or (iv). But the method above is preferable for our present purpose.

Hence, by Art. 10, equation (2),

$$\Delta_0(\lambda) = \begin{vmatrix} a-\lambda, & h, & g, & u \\ h, & b-\lambda, & f, & v \\ g, & f, & c-\lambda, & w \\ u, & v, & w, & d \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$= M^2 \begin{vmatrix} c_1-\lambda, & 0, & 0, & -\lambda p_1 \\ 0, & c_2-\lambda, & 0, & -\lambda p_2 \\ 0, & 0, & c_3-\lambda, & k-\lambda p_3 \\ -\lambda p_1, & -\lambda p_2, & k-\lambda p_3, & c_4-\lambda\Sigma p^2 \end{vmatrix}$$

where M is the determinant of the substitution expressing $(y_1, y_2, y_3, 1)$ in terms of $(x, y, z, 1)$. Now

$$y_1 = x_1 - p_1, \quad y_2 = x_2 - p_2, \quad y_3 = x_3 - p_3,$$

so that M is also equal to the determinant of the substitution expressing (x_1, x_2, x_3) in terms of (x, y, z). Thus, since

$$(a, b, c, f, g, h \between x, y, z)^2 - \lambda(x^2 + y^2 + z^2)$$
$$= (c_1x_1^2 + c_2x_2^2 + c_3x_3^2) - \lambda(x_1^2 + x_2^2 + x_3^2),$$

$$\Delta_1(\lambda) = \begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix} = M^2 \begin{vmatrix} c_1-\lambda, & 0, & 0 \\ 0, & c_2-\lambda, & 0 \\ 0, & 0, & c_3-\lambda \end{vmatrix} \quad \ldots (7).$$

If we equate coefficients of λ^3 in equation (7) we see that* $M^2 = 1$; further we find on division of equation (6) by (7)

$$\frac{\Delta_0(\lambda)}{\Delta_1(\lambda)} = c_4 - \lambda\Sigma p^2 - \frac{\lambda^2 p_1^2}{c_1-\lambda} - \frac{\lambda^2 p_2^2}{c_2-\lambda} - \frac{(k-\lambda p_3)^2}{c_3-\lambda} \quad \ldots\ldots\ldots (8).$$

Let both sides of equation (8) be expanded in ascending powers of λ, which is possible provided that λ is numerically less than the smallest non-zero value of c_1, c_2, c_3. On the left-hand side the first term in the expansion is equal to the quotient of the term of lowest degree in $\Delta_0(\lambda)$ by the term of lowest degree in $\Delta_1(\lambda)$; on the right-hand side the first term is different in each of the three cases.

In case (5_1), $k = 0$; so the first term in the expansion of (8) is c_4, even when one or more of c_1, c_2, c_3 may be zero.

In case (5_2), $k = 0$, $c_4 = 0$; so that the first term of the expansion has a *positive* power of λ.

In case (5_3), $c_3 = 0$; and so the first term in the expansion is k^2/λ.

* Of course this is a well-known property of orthogonal substitutions.

We have thus the following rule* for writing down the reduced forms :—

Factorize $\Delta_1(\lambda)$, *which gives* c_1, c_2, c_3; *and take the quotient of* δ_0, *the term of lowest degree in* $\Delta_0(\lambda)$, *by* δ_1, *the term of lowest degree in* $\Delta_1(\lambda)$; *then three cases arise :—*

First, if δ_0 *and* δ_1 *are of the same degree in* λ *their quotient is equal to* c_4, *and* $k = 0$;

Secondly, if δ_0 *is of higher degree than* δ_1, $c_4 = 0$ *and* $k = 0$;

Thirdly, if δ_0 *is of lower degree than* δ_1, *their quotient is equal to* k^2/λ *and* $c_3 = 0$.

Thus we can classify as follows :—

First type; $\delta_0/\delta_1 = c_4$; the reduced equation is

$$c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2 + c_4 = 0 \quad\quad (5_1),$$

$c_1, c_2, c_3 \neq 0$	Central quadric.
$c_1, c_2 \neq 0$; $c_3 = 0$	Elliptic or Hyperbolic cylinder.
$c_1 \neq 0$; $c_2, c_3 = 0$	Two parallel planes.

Second type; $\delta_0/\delta_1 =$ a positive power of λ; the reduced equation is

$$c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2 = 0 \quad\quad (5_2),$$

$c_1, c_2, c_3 \neq 0$	Cone.
$c_1, c_2 \neq 0$; $c_3 = 0$	Two intersecting planes.
$c_1 \neq 0$; $c_2, c_3 = 0$	Two coincident planes.

Third type; $\delta_0/\delta_1 = k^2/\lambda$; the reduced equation is

$$c_1 y_1^2 + c_2 y_2^2 + 2k y_3 = 0 \quad\quad (5_3),$$

$c_1, c_2 \neq 0$; $c_3 = 0$	Paraboloid.
$c_1 \neq 0$; $c_2, c_3 = 0$	Parabolic cylinder.

CONICS.

For the conic $\quad (a, b, c, f, g, h \between x, y, 1)^2 = 0 \quad\quad (9),$

we write $\quad \Delta_1(\lambda) = (a - \lambda)(b - \lambda) - h^2 = (c_1 - \lambda)(c_2 - \lambda),$

$$\Delta_0(\lambda) = \begin{vmatrix} a - \lambda, & h, & g \\ h, & b - \lambda, & f \\ g, & f, & c \end{vmatrix},$$

* The advantage of this rule is that it gives one definite process in all cases; in the ordinary methods, each class has to be treated in a different way.

and as before we denote by δ_0, δ_1, the terms of lowest degree in $\Delta_0(\lambda)$, $\Delta_1(\lambda)$, respectively.

Then the classification is :—

First type; $\delta_0/\delta_1 = c_3$; the reduced equation is

$$c_1 y_1^2 + c_2 y_2^2 + c_3 = 0 \quad\ldots\ldots\ldots\ldots(10_1),$$

$c_1, c_2 \neq 0$	Ellipse or Hyperbola.
$c_1 \neq 0;\ c_2 = 0$	Two parallel lines.

Second type; $\delta_0/\delta_1 =$ a positive power of λ; the reduced equation is

$$c_1 y_1^2 + c_2 y_2^2 = 0 \quad\ldots\ldots\ldots\ldots(10_2),$$

$c_1, c_2 \neq 0$	Two intersecting lines.
$c_1 \neq 0;\ c_2 = 0$	Two coincident lines.

Third type; $\delta_0/\delta_1 = k^2/\lambda$; the reduced equation is

$$c_1 y_1^2 + 2k y_2 = 0 \quad\ldots\ldots\ldots\ldots(10_3),$$

$c_1 \neq 0;\ c_2 = 0$	Parabola.

30. Examples to illustrate the methods of the last article.

Example 1. *Consider the conic*

$$x^2 + 4xy + 4y^2 + 4x + 6y + 7 = 0.$$

Here
$$\Delta_1(\lambda) = \begin{vmatrix} 1-\lambda, & 2 \\ 2, & 4-\lambda \end{vmatrix} = \lambda(\lambda-5),$$

and
$$\Delta_0(\lambda) = \begin{vmatrix} 1-\lambda, & 2, & 2 \\ 2, & 4-\lambda, & 3 \\ 2, & 3, & 7 \end{vmatrix} = -1 + \text{terms in } \lambda.$$

Thus the quotient δ_0/δ_1 is $1/5\lambda$; consequently the conic is a *parabola* and its reduced form is

$$5x_1^2 + 2x_2/\sqrt{5} = 0,$$

so that the latus-rectum is $2/5^{\frac{3}{2}}$.

Example 2. Let us find *the conditions that a quadric may be a surface of revolution.*

This requires that two of the roots c_1, c_2, c_3 shall be equal; say $c_2 = c_1$. Then since the invariant-factors of $\Delta_1(\lambda)$ are *linear* (Art. 26) the factor $(\lambda - c_1)$ must divide every first minor in

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix}.$$

Thus we must have

$$gh-(a-c_1)f=0, \quad hf-(b-c_1)g=0, \quad fg-(c-c_1)h=0 \quad\text{.........(1).}$$

Or, if none of f, g, h are zero,

$$c_1=a-gh/f=b-hf/g=c-fg/h \quad\text{........................(2).}$$

When these conditions are satisfied the other three minors, such as

$$(b-c_1)(c-c_1)-f^2,$$

are also zero; and so the conditions (2) are both necessary and sufficient.

But if f, say, is zero we must have $gh=0$, so that g or h must also be zero; take $f=0$, $g=0$, $h\neq 0$. Then we get $c-c_1=0$, and from the other minors

$$(a-c)(b-c)-h^2=0 \quad\text{.............................. (3).}$$

If all of f, g, h are zero, two of the numbers a, b, c must be equal.

Example 3. *The determination of the form of a plane section of a quadric.*

Take the quadric

$$(a, b, c, f, g, h\between x, y, z)^2+2(ux+vy+wz)+d=0,$$

and the plane $$lx+my+nz+p=0,$$

where we shall assume for simplicity of statement that $l^2+m^2+n^2=1$.

If now we write $$x_3=lx+my+nz+p,$$

we can introduce a set of orthogonal coordinates x_1, x_2, x_3, such that x_1, x_2 are measured along any rectangular axes in the plane of the section.

Suppose that the equation to the quadric becomes

$$\Sigma a_{rs}x_rx_s+2\Sigma u_rx_r+t=0, \qquad (r, s=1, 2, 3),$$

the conic to be discussed is thus

$$a_{11}x_1^2+2a_{12}x_1x_2+a_{22}x_2^2+2(u_1x_1+u_2x_2)+t=0,$$

and the two fundamental determinants of Art. 29 are

$$\Delta_1(\lambda)=\begin{vmatrix} a_{11}-\lambda, & a_{12} \\ a_{21}, & a_{22}-\lambda \end{vmatrix}, \quad \Delta_0(\lambda)=\begin{vmatrix} a_{11}-\lambda, & a_{12}, & u_1 \\ a_{21}, & a_{22}-\lambda, & u_2 \\ u_1, & u_2, & t \end{vmatrix}.$$

But these are the minors of $a_{33}-\lambda$ in the two determinants

$$\begin{vmatrix} a_{11}-\lambda, & a_{12}, & a_{13} \\ a_{21}, & a_{22}-\lambda, & a_{23} \\ a_{31}, & a_{32}, & a_{33}-\lambda \end{vmatrix} \text{ and } \begin{vmatrix} a_{11}-\lambda, & a_{12}, & a_{13}, & u_1 \\ a_{21}, & a_{22}-\lambda, & a_{23}, & u_2 \\ a_{31}, & a_{32}, & a_{33}-\lambda, & u_3 \\ u_1, & u_2, & u_3, & t \end{vmatrix}.$$

Thus, since $$\Sigma a_{rs}x_rx_s-\lambda\Sigma x_r^2+2\Sigma u_rx_r+t \qquad (r, s=1, 2, 3)$$

is a transformation of

$$(a, b, c, f, g, h\between x, y, z)^2-\lambda(x^2+y^2+z^2)+2(ux+vy+wz)+d,$$

with $$x_3=lx+my+nz+p,$$

it follows from Art. 10 equation (4) that

$$-\Delta_1(\lambda)=\begin{vmatrix} a-\lambda, & h, & g, & l \\ h, & b-\lambda, & f, & m \\ g, & f, & c-\lambda, & n \\ l, & m, & n, & 0 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and

$$-\Delta_0(\lambda)=\begin{vmatrix} a-\lambda, & h, & g, & u, & l \\ h, & b-\lambda, & f, & v, & m \\ g, & f, & c-\lambda, & w, & n \\ u, & v, & w, & d, & p \\ l, & m, & n, & p, & 0 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

From (4) and (5) we can obtain the reduced forms as in Art. 29.

To illustrate still further, let us take a central surface referred to its principal axes, so that

$$f=0, \quad g=0, \quad h=0, \quad u=0, \quad v=0, \quad w=0.$$

Then we find

$$\Delta_1(\lambda)=\Sigma l^2(b-\lambda)(c-\lambda)=\Sigma l^2 bc-\lambda\Sigma l^2(b+c)+\lambda^2,$$

$$\Delta_0(\lambda)=d\,.\,\Delta_1(\lambda)+p^2(a-\lambda)(b-\lambda)(c-\lambda).$$

Thus, assuming no special relation between the plane and the surface we get the reduced equation

$$c_1 y_1^2+c_2 y_2^2+c_3=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6_1),$$

where c_1, c_2 are roots of $\Delta_1(\lambda)=0$ and

$$c_3=d+abcp^2/\Sigma l^2 bc \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6_2).$$

If the section is parabolic, we must have $\Sigma l^2 bc=0$, or $\Sigma l^2/a=0$ and then

$$\Sigma l^2(b+c)=(bc+ca+ab)\,\Sigma l^2/a-abc\Sigma l^2/a^2=-abc\Sigma l^2/a^2.$$

Thus from $\Delta_1(\lambda)=0$ we get $\quad c_1=-abc\Sigma l^2/a^2, \quad c_2=0,$

and $$\delta_0=p^2abc, \quad \delta_1=-\lambda\Sigma l^2(b+c)=\lambda abc\Sigma l^2/a^2,$$

so that $$\frac{\delta_0}{\delta_1}=\frac{p^2abc}{abc(\Sigma l^2/a^2)\lambda}=\frac{1}{\lambda}\frac{p^2}{\Sigma l^2/a^2}.$$

Thus the reduced equation is

$$-abc(\Sigma l^2/a^2)\,y_1^2+\frac{2py_2}{(\Sigma l^2/a^2)^{\frac{1}{2}}}=0 \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

so that the latus-rectum is $\quad \dfrac{2p}{abc}(\Sigma l^2/a^2)^{-\frac{3}{2}}.$ (Math. Trip. 1889 and 1898.)

Example 4. *Circular sections of a quadric referred to its principal axes.*

Here we have $\quad f=0, \quad g=0, \quad h=0, \quad u=0, \quad v=0, \quad w=0.$

If $\Delta_1(\lambda)$ (equation (4) above) is to have equal roots all the first minors

(except those arising from the last row or column) will have a common factor; thus the six minors

$$(b-\lambda)\,n^2+(c-\lambda)\,m^2,\ \text{etc.},\quad (a-\lambda)\,mn,\ \text{etc.}$$

have a common factor. This leads at once to the three solutions

$$\begin{array}{llll} \text{(i)} & a-\lambda=0, & l=0, & (a-b)\,n^2+(a-c)\,m^2=0, \\ \text{(ii)} & b-\lambda=0, & m=0, & (b-a)\,l^2+(b-a)\,n^2=0, \\ \text{(iii)} & c-\lambda=0, & n=0, & (c-a)\,m^2+(c-b)\,l^2=0. \end{array}$$

Of these only one can give real sections; this one is the second, if a, b, c are arranged in order of magnitude; and then

$$\frac{l^2}{a-b}=\frac{n^2}{b-c}=\frac{1}{a-c}\quad\text{..............................(8).}$$

Thus $\qquad \Delta_1(\lambda)=(b-\lambda)^2,$

and $\qquad \Delta_0(\lambda)=d\,(b-\lambda)^2+p^2\,(a-\lambda)\,(b-\lambda)\,(c-\lambda),$

so that the quotient δ_0/δ_1 is $d+p^2ac/b$, and the equation to the circle is

$$b\,(y_1{}^2+y_2{}^2)+(d+p^2ac/b)=0\quad\text{..........................(9).}$$

Example 5. Find the angle θ between the circular sections of the quadric

$$2fyz+2gzx+2hxy+d=0. \qquad \text{(Math. Trip. 1905.)}$$

If this equation is reduced to

$$c_1y_1{}^2+c_2y_2{}^2+c_3y_3{}^2+d=0, \qquad (c_1>c_2>c_3),$$

the angle θ is given by $\tan^2\frac{1}{2}\theta=(c_1-c_2)/(c_2-c_3)$; in virtue of equation (8).

Thus, if $\cos\theta=t$, $\qquad t=(c_1-2c_2+c_3)/(c_1-c_3)$ ....(10).

Now c_1, c_2, c_3 are roots of the equation

$$\begin{vmatrix} -\lambda, & h, & g \\ h, & -\lambda, & f \\ g, & f, & -\lambda \end{vmatrix}=0,$$

so that $\qquad \Sigma c_1=0,\quad \Sigma c_2c_3=-\Sigma f^2,\quad c_1c_2c_3=2fgh.$

From the two equations

$$c_1+c_2+c_3=0,\quad c_1\,(1-t)-2c_2+c_3\,(1+t)=0,$$

we get $\qquad c_1=-(3+t)\,\rho,\quad c_2=2t\rho,\quad c_3=(3-t)\,\rho,$

where ρ is a factor of proportionality. Hence we find

$$\left.\begin{aligned} \rho^2\,(9+3t^2)&=-\Sigma c_2c_3=+\Sigma f^2, \\ -\rho^3t\,(9-t^2)&=\tfrac{1}{2}c_1c_2c_3=\ \ fgh \end{aligned}\right\}\quad\text{....................(11).}$$

Thus $\cos\theta$ is a root of

$$\left(\frac{9+3t^2}{\Sigma f^2}\right)^3=\frac{t^2\,(9-t^2)^2}{(fgh)^2}\quad\text{..........................(12).}$$

The six roots of equation (12) are found from (10) by permuting c_1, c_2, c_3 in pairs; and since c_1, c_2, c_3 are real it follows that all the roots will be real: but only two of them lie between -1 and $+1$, and these two are equal and opposite. There are therefore only two real values of θ, which are supplementary.

31. Conics and Quadrics (in tangential coordinates).

If a quadric is given by its tangential equation (in plane-coordinates) its reduced form can usually* be found by transformation to point-coordinates and then applying the method of Art. 29. As a matter of fact, however (quite apart from the special cases which would be excluded by this method), the work would be then considerably longer than is necessary. It is therefore worth while to give a direct method of treatment, particularly as in this form the problem is rather more directly connected with the general methods of Chapter III, than in point-coordinates. (Compare Lindemann in his edition of Clebsch's *Vorlesungen über Geometrie*, Bd. 2, pp. 254—266.)

Let x_1, x_2, x_3 be a set of orthogonal Cartesian coordinates, u_1, u_2, u_3, u_4 being the associated plane coordinates, so that

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4 = 0$$

is the equation to a typical plane.

Then in transforming to another set of orthogonal axes, the x's are subjected to a linear substitution, say

$$\left.\begin{aligned} x_1 &= l_{11}y_1 + l_{12}y_2 + l_{13}y_3 + l_{14} \\ x_2 &= l_{21}y_1 + l_{22}y_2 + l_{23}y_3 + l_{24} \\ x_3 &= l_{31}y_1 + l_{32}y_2 + l_{33}y_3 + l_{34} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1);$$

and thus the u's are changed into v's such that†

$$\left.\begin{aligned} v_1 &= l_{11}u_1 + l_{21}u_2 + l_{31}u_3 \\ v_2 &= l_{12}u_1 + l_{22}u_2 + l_{32}u_3 \\ v_3 &= l_{13}u_1 + l_{23}u_2 + l_{33}u_3 \\ v_4 &= l_{14}u_1 + l_{24}u_2 + l_{34}u_3 + u_4 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Further, because the expression

$$(u_1x_1 + u_2x_2 + u_3x_3 + u_4)/(u_1^2 + u_2^2 + u_3^2)^{\frac{1}{2}}$$

represents the length of the perpendicular from (x_1, x_2, x_3) on the plane (u_1, u_2, u_3, u_4), it follows that

$$A = u_1^2 + u_2^2 + u_3^2 = v_1^2 + v_2^2 + v_3^2 \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

* Cases of exception occur when the quadric degenerates into a conic.

† Because $u_4 + \Sigma ux \equiv v_4 + \Sigma vy$.

since the perpendicular is obviously independent of the particular set of axes employed. Hence the substitution connecting u_1, u_2, u_3 with v_1, v_2, v_3 must be orthogonal; and its determinant is therefore ± 1 (see Art. 29, p. 73).

Again, the determinant of the substitution (2) connecting v_1, v_2, v_3, v_4 with u_1, u_2, u_3, u_4 is

$$\begin{vmatrix} l_{11}, & l_{21}, & l_{31}, & 0 \\ l_{12}, & l_{22}, & l_{32}, & 0 \\ l_{13}, & l_{23}, & l_{33}, & 0 \\ l_{14}, & l_{24}, & l_{34}, & 1 \end{vmatrix} = \begin{vmatrix} l_{11}, & l_{21}, & l_{31} \\ l_{12}, & l_{22}, & l_{32} \\ l_{13}, & l_{23}, & l_{33} \end{vmatrix} = \pm 1 \quad \text{.........(4)},$$

according to what we have just proved.

Now if the quadric has the tangential equation

$$B = \sum_{r,s=1}^{4} b_{rs} u_r u_s = 0 \quad \text{........................(5)},$$

where $b_{rs} = b_{sr}$, it follows from Art. 10 that in the linear substitution (2) the determinant $|B - \lambda A|$ is multiplied by the square of the determinant (4); and *consequently the determinant* $|B - \lambda A|$ *is invariant under the substitution* (2).

Now, regarded as a form in four variables, A is semi-definite and its rank is 3; so that, by Art. 27, the determinant $|\lambda A + \mu B|$ may have one invariant-factor equal to μ, or it may have one equal to μ^2; and in the latter case, the coefficient b_{44} must be zero. The remaining invariant-factors are of course linear. Thus we have the two cases:—

First, $b_{44} \neq 0$, one invariant-factor μ; the reduced forms are

$$A = v_1^2 + v_2^2 + v_3^2, \quad B = c_1 v_1^2 + c_2 v_2^2 + c_3 v_3^2 + c_4 v_4^2 \quad \text{......}(6_1),$$

where c_1, c_2, c_3, c_4 are given by*

$$c_4 (c_1 - \lambda)(c_2 - \lambda)(c_3 - \lambda) = |B - \lambda A| \quad \text{..............}(6_2),$$

so that $c_4 = b_{44}$.

Secondly, $b_{44} = 0$, one invariant-factor μ^2; the reduced forms are

$$A = v_1^2 + v_2^2 + v_3^2, \quad B = c_1 v_1^2 + c_2 v_2^2 + 2k v_3 v_4 \quad \text{.........}(7_1),$$

where c_1, c_2, k are given by†

$$-k^2 (c_1 - \lambda)(c_2 - \lambda) = |B - \lambda A| \quad \text{.................}(7_2),$$

so that $$k^2 = b_{14}^2 + b_{24}^2 + b_{34}^2.$$

* Lindemann, *l.c.*, p. 255, equation (3*a*) and p. 258, equation (14).

† Lindemann, *l.c.*, p. 260, equation (16) and p. 261 (foot).

It is now easy to make a metrical classification of all quadrics given by a tangential equation :—

(i)	$c_4=b_{44}\neq 0$	$\Sigma c_r v_r^2=0 \quad (r=1, 2, 3, 4)$
	$c_1, c_2, c_3\neq 0$	Central quadric
	$c_1, c_2\neq 0;\ c_3=0$	Central conic
	$c_1\neq 0;\ c_2, c_3=0$	Two points
	$c_1, c_2, c_3=0$	Two coincident points
(ii)	$b_{44}=0$	$c_1v_1^2+c_2v_2^2+2kv_3v_4=0$
	$c_1, c_2\neq 0$	Paraboloid
	$c_1\neq 0;\ c_2=0$	Parabola
	$c_1, c_2=0$	Two points ; one at infinity

We have excluded the case in which $|B-\lambda A|$ is identically zero: for then $B=0$ will be a conic in the plane at infinity.

The corresponding classification for conics is obtained by rejecting the first possibility in each of the two classes. As an exercise, the reader may obtain the results of Exs. 3—5, Art. 30, using tangential equations.

32. To illustrate the method of the last article, we take two conics ; the work for quadrics is simply heavier without being essentially different.

(i) $$5u_1^2+5u_2^2-u_3^2+2u_2u_3+4u_3u_1-8u_1u_2=0.$$

Here
$$\begin{vmatrix} 5-\lambda, & -4, & 2 \\ -4, & 5-\lambda, & 1 \\ 2, & 1, & -1 \end{vmatrix} = -(5-\lambda)(10-\lambda)$$
is the working determinant, thus*
$$c_1=5, \quad c_2=10, \quad c_3=-1,$$
and the reduced form is
$$5v_1^2+10v_2^2-v_3^2=0,$$
representing an ellipse of semi-axes $\surd 5$, $\surd 10$.

(ii) $$5u_1^2+2u_2^2-2u_2u_3+4u_3u_1+6u_1u_2=0.$$

Here the determinant is
$$\begin{vmatrix} 5-\lambda, & 3, & 2 \\ 3, & 2-\lambda, & -1 \\ 2, & -1, & 0 \end{vmatrix} = 5(\lambda-5).$$

* Of course in working with conics u_3, c_3 correspond to u_4, c_4 of the work with quadrics.

Hence $c_1 = 5$, $k^2 = 5$; and the reduced form is

$$5v_1^2 + 2\sqrt{5}\, v_2 v_3 = 0,$$

or

$$\sqrt{5}\, v_1^2 + 2v_2 v_3 = 0,$$

representing a parabola of latus-rectum $2\sqrt{5}$.

33. In dealing with conics and quadrics other problems present themselves which can be solved by the methods explained in Chapters I—IV*. We may mention the metrical classification of conics and quadrics expressed by means of equations in homogeneous coordinates. For details of the work the reader may consult a paper by the author†; but it may be sufficient to say that if the point-coordinates x_1, x_2, x_3 are areals and the equation to a conic is

$$\Sigma a_{rs} x_r x_s = 0, \qquad (r, s = 1, 2, 3),$$

then the determinants (6) and (7) of Art. 29 are replaced by

$$\Delta_0(\lambda) = \begin{vmatrix} a_{11} - \lambda k_1, & a_{12}, & a_{13} \\ a_{21}, & a_{22} - \lambda k_2, & a_{23} \\ a_{31}, & a_{32}, & a_{33} - \lambda k_3 \end{vmatrix},$$

and

$$\Delta_1(\lambda) = - \begin{vmatrix} a_{11} - \lambda k_1, & a_{12}, & a_{13}, & 1 \\ a_{21}, & a_{22} - \lambda k_2, & a_{23}, & 1 \\ a_{31}, & a_{32}, & a_{33} - \lambda k_3, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix},$$

where $\quad k_1 = bc\cos A, \quad k_2 = ca\cos B, \quad k_3 = ab\cos C.$

In tangential coordinates u_1, u_2, u_3, the determinant replacing $|B - \lambda A|$ in equations (6_2), (7_2), Art. 31, is

$$\frac{1}{16\Delta^4} \begin{vmatrix} a_{11} - \lambda a^2, & a_{12} + \lambda ab\cos C, & a_{13} + \lambda ca\cos B \\ a_{21} + \lambda ab\cos C, & a_{22} - \lambda b^2, & a_{23} + \lambda bc\cos A \\ a_{31} + \lambda ca\cos B, & a_{32} + \lambda bc\cos A, & a_{33} - \lambda c^2 \end{vmatrix},$$

where Δ is the area of the triangle of reference.

Another problem is the metrical classification of conics and quadrics in elliptic and hyperbolic (non-Euclidean) space; for details the reader may consult another paper by the author‡. It may be useful to add, that in hyperbolic space, the classification is algebraically the same as the classification of bicircular quartics given in the following article:

* The reader will find many applications in Lindemann's edition of Clebsch's *Vorlesungen über Geometrie*, Bd. 2. pp. 215—388.

† *Proc. Camb. Phil. Soc.*, vol. x, 1900, p. 349. Some differences of notation from the above will be found in this paper.

‡ *Trans. Amer. Math. Soc.*, vol. VI, July, 1905, p. 275.

but there are no cases to be rejected as representing degenerate surfaces (see the note at the foot of this page).

34. Bicircular quartics (or cyclic curves).

These quartics have nodes at the circular points, and consequently their equations are of the type

$$(x^2+y^2)^2 + 2(x^2+y^2)(lx+my) + (a, b, c, f, g, h\between x, y, 1)^2 = 0 \ldots(1),$$

referred to ordinary Cartesian coordinates. The coefficients in (1) will be supposed real, and we shall introduce z to make the equation homogeneous, and shall then write $x^2+y^2=2tz$, so that (1) becomes

$$4t^2 + 4t(lx+my) + (a, b, c, f, g, h\between x, y, z)^2 = 0 \ldots\ldots\ldots(2).$$

We are thus led to discuss the two quadratic forms

$$\left.\begin{aligned} A &= x^2+y^2-2tz, \\ B &= (a, b, c, f, g, h\between x, y, z)^2 + 4t(lx+my) + 4t^2 \end{aligned}\right\} \ldots\ldots\ldots(3).$$

The determinant $|B-\lambda A|$ is

$$\begin{vmatrix} -\lambda+a, & h, & g, & 2l \\ h, & -\lambda+b, & f, & 2m \\ g, & f, & c, & +\lambda \\ 2l, & 2m, & +\lambda, & 4 \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and, since the signature of A is 2, this determinant has at least two real invariant-factors of odd degree, in consequence of Klein's theorem (Art. 28).

Thus, from the general list of types for four variables, we must omit the cases [22], [(22)], [4] and the singular case (because $|A|=-1$); and of course [(1111)] is geometrically superfluous. Further, there can be two cases only with complex invariant factors, namely $[11\bar{1}\bar{1}]$ and $[(11)\,\bar{1}\bar{1}]$, where the bar indicates a pair of conjugate complex roots.

The following are the only cases of interest from a geometrical point of view*:—

[1111] *The general bicircular quartic with two branches (and four concyclic real foci).* The reduced equation is

$$\left.\begin{aligned} c_1X_1^2 + c_2X_2^2 + c_3X_3^2 - c_4X_4^2 &= 0 \\ \text{where}\qquad X_1^2 + X_2^2 + X_3^2 - X_4^2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots(5_1),$$

and $(\lambda-c_1)$, $(\lambda-c_2)$, $(\lambda-c_3)$, $(\lambda-c_4)$ are the factors of (4).

* If $(\lambda-c)$ is a factor of every first minor in (4), the form $(cA-B)$ contains only two variables, and so factorizes; these cases are indicated by the presence of round brackets in the symbols of Art. 18; the quartic then degenerates into two circles.

[11 $\overline{11}$] *The general bicircular quartic with one branch (and four real foci which are not concyclic).* The reduced equation is

$$\left.\begin{aligned} c_1X_1^2 + c_2X_2^2 + \alpha(X_3^2 - X_4^2) + 2\beta X_3X_4 &= 0 \\ \text{where} \qquad X_1^2 + X_2^2 + X_3^2 - X_4^2 &= 0 \end{aligned}\right\} \ldots(5_2),$$

and $(\lambda - c_1)$, $(\lambda - c_2)$, $[(\lambda - \alpha)^2 + \beta^2]$ are the factors of (4).

[211] *The nodal bicircular quartic (e.g. a limaçon*)*; the reduced equation is

$$\left.\begin{aligned} c_1X_1^2 + c_2X_2^2 - 2c_3X_3X_4 + kX_3^2 &= 0 \\ \text{where} \qquad X_1^2 + X_2^2 - 2X_3X_4 &= 0 \end{aligned}\right\} \ldots\ldots\ldots(5_3),$$

and $(\lambda - c_1)$, $(\lambda - c_2)$, $(\lambda - c_3)^2$ are the factors of (4).

This quartic is the inverse of a central conic.

[31] *The cuspidal bicircular quartic (e.g. a cardioid*)*; the reduced equation is

$$\left.\begin{aligned} c_1X_1^2 + c_2(X_2^2 - 2X_3X_4) + 2X_2X_3 &= 0 \\ \text{where} \qquad X_1^2 + X_2^2 - 2X_3X_4 &= 0 \end{aligned}\right\} \ldots\ldots\ldots(5_4),$$

and $(\lambda - c_1)$, $(\lambda - c_2)^3$ are the factors of (4).

This quartic is the inverse of a parabola.

The equations

$$X_1 = 0,\ X_2 = 0,\ X_3 = 0,\ X_4 = 0$$

represent circles or straight lines; the first form of the identity (5_1 and 5_2) implies that the four circles are mutually orthogonal, the first three being real†: in the second identity X_3 and X_4 are point-circles at the intersections of X_1 and X_2 (which are orthogonal).

For details as to the theory of bicircular quartics the reader may consult Salmon's *Higher Plane Curves* (Arts. 269—282) and Casey's paper (*Trans. Royal Irish Academy*, vol. XXIV, 1869, p. 457). However, the foregoing theory is more closely associated with the discussion of Cyclides in Salmon's *Geometry of Three Dimensions* (Arts. 562—566). The geometrical interpretation of the algebraic transformations used above will be found in Chapter II of Bôcher's book quoted in the next article: in spite of its elegance, an account of this would occupy too much space here.

* The limaçon and cardioid have cusps at the circular points: but this does not lead to any special feature from our present point of view. By inversion these cusps can be reduced to nodes.

† The fourth has real coefficients, but does not pass through any real points; for instance, $x^2 + y^2 + 1 = 0$.

By inversion and change of axes, the four circles X can be reduced to the symmetrical types

$$X_1=x,\quad X_2=y,\quad X_3=\tfrac{1}{2}(x^2+y^2-1),\quad X_4=\tfrac{1}{2}(x^2+y^2+1)\ \ldots(5_1 \text{ and } 5_2),$$

or $$X_1=x,\quad X_2=y,\quad X_3=\tfrac{1}{2}(x^2+y^2),\qquad X_4=1\ \ldots\ldots\ldots\ldots(5_3 \text{ and } 5_4).$$

In the former case we can associate points on the plane with points on the sphere $\xi^2+\eta^2+\zeta^2=1$, by writing

$$\xi=X_1/X_4,\quad \eta=X_2/X_4,\quad \zeta=X_3/X_4;$$

and this transformation is equivalent to a stereographic projection (or inversion) from the vertex $(0, 0, 1)$. Then the quartics are the projections of the intersections of the sphere with

either the cone $\qquad (c_1-c_4)\,\xi^2+(c_2-c_4)\,\eta^2+(c_3-c_4)\,\zeta^2=0\ \ldots\ldots\ldots\ldots(5_1),$

or the paraboloid $\qquad (c_1-a)\,\xi^2+(c_2-a)\,\eta^2+2\beta\zeta=0\ \ldots\ldots\ldots\ldots(5_2).$

From this projection it is at once evident that the curves (5_1) consist of two ovals; and (5_2) of one oval only.

The quartics confocal to (5_1) are given by the confocal cones

$$\xi^2/(p+\lambda)+\eta^2/(q+\lambda)+\zeta^2/(r+\lambda)=0,$$

where $$p^{-1}=c_1-c_4,\quad q^{-1}=c_2-c_4,\quad r^{-1}=c_3-c_4;$$

and the quartics intersect orthogonally, because the cones cut orthogonally. The 16 foci of these quartics lie by fours on the circles X_1, X_2, X_3, X_4; and the quartics are their own inverses for these circles. In the symmetrical case, the four *real* foci are collinear and are given by

$$(p-q)^{\frac{1}{2}}x=\pm(p-r)^{\frac{1}{2}}\pm(q-r)^{\frac{1}{2}},\qquad (p>q>r).$$

Figures of the symmetrical set are to be found on p. 133 of A. C. Dixon's *Elliptic Functions* and on p. 72 of Bôcher's book quoted in Art. 35. If this set is inverted we get the general type of confocal quartics for which the real foci lie on one of the circles of inversion (Bôcher, p. 74).

The 16 foci of (5_2) lie on the circles X_1, X_2, $X_3\pm iX_4$; and by using the circles of reference $X_3\pm iX_4$ (instead of X_3, X_4) the confocals can be brought to the same algebraic type as those of (5_1). The four *real* foci in the symmetrical case are given by

$$(c_1-c_2)\,\beta\,(Z^2-Z^{-2})=2\left[(c_1-a)(c_2-a)+\beta^2\right],\qquad Z=x+iy,$$

two being on each of the axes of reference. It may be noted that the real foci are generally not concyclic, but will lie on the circle X_3 if it happens that

$$(c_1-a)(c_2-a)+\beta^2=0.$$

The real foci are then *harmonically* situated on their circle which, of course, is *not* a circle of inversion. Figures of these confocals are given by Dixon on p. 134, and by Bôcher on p. 73; in Bôcher's figures the foci are concyclic.

The confocals of (5_3) are given by

$$X_1^2/(p+\lambda)+X_2^2/(q+\lambda)+X_3^2=0,$$

where $$kp^{-1}=c_1-c_3, \quad kq^{-1}=c_2-c_3;$$

and the two real foci in the symmetrical case are given by

$$4/x^2=q-p, \quad y=0, \qquad \text{(if } q>p).$$

By projecting the spherical figures of (5_1) and (5_2) from a focus as vertex we obtain sets of confocal Cartesian ovals. It is easy to prove that the confocal set with foci on $y=0$ at $x=0, 1, c, \infty$ is given by

$$X_1^2/\lambda+X_2^2/(\lambda-c)=X_4^2/(\lambda-1), \qquad X_1^2+X_2^2+X_3^2=X_4^2,$$

where $$c^{\frac{1}{2}}X_1=\tfrac{1}{2}(x^2+y^2-c), \quad (c^2-c)^{\frac{1}{2}}X_2=\tfrac{1}{2}(x^2+y^2-2cx+c),$$

$$X_3=y, \quad (c-1)^{\frac{1}{2}}X_4=\tfrac{1}{2}(x^2+y^2-2x+c).$$

An interesting set of bicircular quartics which have *two* foci ($x=\pm 1$, $y=0$) in common, but are *not* confocal, is the set of Cassinian ovals

$$(x^2+y^2)^2-2(x^2-y^2)=c^2-1,$$

which belong to the type (5_1) if $c^2<1$, to (5_2) if $c^2>1$, and to (5_3) if $c^2=1$. The general form of these curves is well known (see for example Edward's *Differential Calculus*, Art. 458).

35. Cyclides*.

These surfaces in space are analogous to bicircular quartics in a plane; the imaginary circle at infinity is a nodal curve on the surface. The equation to a cyclide in rectangular Cartesian coordinates x_1, x_2, x_3 is of the form

$$(x_1^2+x_2^2+x_3^2)^2+2(x_1^2+x_2^2+x_3^2)(l_1x_1+l_2x_2+l_3x_3)x_4+x_4^2\sum_{r,s=1}^{4}a_{rs}x_rx_s=0,$$

where $x_4(=1)$ is inserted to make the equation homogeneous.

Thus if we write $$x_1^2+x_2^2+x_3^2=2x_4x_5,$$

we get $$\sum_{r,s=1}^{4}a_{rs}x_rx_s+4(l_1x_1+l_2x_2+l_3x_3)x_5+4x_5^2=0,$$

as the equation. If now we discuss the two quadratic forms

$$\left.\begin{aligned} A &= x_1^2+x_2^2+x_3^2-2x_4x_5 \\ B &= \sum_{r,s=1}^{4}a_{rs}x_rx_s+4(l_1x_1+l_2x_2+l_3x_3)x_5+4x_5^2 \end{aligned}\right\} \quad \ldots\ldots\ldots(1),$$

* For a fuller discussion of the classification of cyclides, see Segre, *Math. Annalen*, Bd. 24, 1884, pp. 313—444.

the determinant $B - \lambda A$ is seen to be

$$\begin{vmatrix} a_{11}-\lambda, & a_{12}, & a_{13}, & a_{14}, & 2l_1 \\ a_{21}, & a_{22}-\lambda, & a_{23}, & a_{24}, & 2l_2 \\ a_{31}, & a_{32}, & a_{33}-\lambda, & a_{34}, & 2l_3 \\ a_{41}, & a_{42}, & a_{43}, & a_{44}, & \lambda \\ 2l_1, & 2l_2, & 2l_3, & \lambda, & 4 \end{vmatrix} \quad \ldots\ldots\ldots\ldots (2).$$

In virtue of Klein's theorem (Art. 28), this determinant has at least three real invariant-factors of odd degree, because the signature of A is 3; and since $|A| = -1$, the determinant (2) cannot vanish identically.

Thus, from the list given in Art. 23, we must reject all characteristics of the types [221], [32], [41], [5], and the singular cases [{3}11], [{3}2], {5}, making 13 cases to be omitted in all. But we may have a pair of conjugate complex roots, represented by [111 11], which gives 3 cases (by adding round brackets where possible); thus we have in all 20 cases (omitting [(11111)]). Of these, the cases of geometrical interest are

$$[11111],\quad [111\,\overline{11}],\quad [(11)\,111],\quad [(11)\,\overline{11}\,1],\quad [(11)(11)\,1],$$
$$[2111],\quad [2\,(11)\,1],\quad [(21)\,11],\quad [(21)\,(11)],$$
$$[311],\quad [3\,(11)],\quad [(31)\,1];$$

since the other 8 cases consist only of combinations of spheres, planes and points*.

We shall not go further into the exhibition of reduced forms and simply refer, for fuller details and diagrams, to Chapter IV of Bôcher's book, *Die Reihentwickelungen der Potentialtheorie* (Leipzig, 1894)†. The geometrical theory is developed in Arts. 560—567 of Salmon's *Geometry of Three Dimensions.* See also Casey's memoir (*Phil. Trans.* vol. CLXI, 1871, p. 585) and Darboux's monograph (*Sur une classe remarquable de courbes et de surfaces algébriques*; Paris, 1873 and 1896).

* It will be seen that in each of these 8 cases, the second minors of (2) have a common factor $(\lambda - c)$, so that $(cA - B)$ contains only two variables and so breaks up into linear factors. Compare notes on pp. 83 and 89.

† Mr J. Fraser has discussed the reduction both for cyclides and bicircular quartics in the case of unequal roots (*Proc. Royal Irish Acad.* vol. XXIV, section A, pp. 71—84, 1904). His method leads to the same determinants as (2) above and (4) of Art. 34; but it would require considerable modification to give the reduction when two roots of the determinant are equal.

It may, however, be remarked that all the cases except the first two [11111], [111 $\overline{11}$] can be obtained by inverting quadric surfaces. Thus we have the table :—

Characteristic	Inverse surface	Other nodes*
[(11) 111] [(11) $\overline{11}$ 1]	Cone	2
[(11) (11) 1]	Cone of revolution	4
[2111]	Ellipsoid or Hyperboloid	1
[2 (11) 1]	Ellipsoid or Hyperboloid of revolution	3
[(21) 11]	Elliptic or Hyperbolic cylinder	1
[(21) (11)]	Circular cylinder	3
[311]	Paraboloid	1
[3 (11)]	Paraboloid of revolution	3
[(31) 1]	Parabolic cylinder	1

The third case [(11) (11) 1] which is found by inverting a circular cone has some special points of interest; it is known as *Dupin's Cyclide*. This surface is met with in many comparatively elementary problems and may be defined in several simple ways; for example, it is the envelope of a sphere whose centre lies in a fixed plane and which touches two fixed spheres (cf. Salmon, *l.c.* Art. 567). From this definition, taking the fixed plane perpendicular to the line of centres of the fixed spheres, we see that the *anchor-ring* or *tore* is a Dupin's cyclide; it would seem at first sight that the tore is a special type, but this is not so from the algebraic point of view; and as a matter of fact the tore can be inverted into the general Dupin's cyclide.

Dupin's cyclide occurs in the solution of Maxwell's problem of finding two curves such that the congruence of lines meeting the curves can be cut orthogonally by a family of surfaces (*Quarterly Journal of Mathematics*, vol. IX, 1867, pp. 111—126). As is now well known, the two curves must be conics in perpendicular planes, related in the same manner as focal conics of a confocal system of quadrics; and the surfaces orthogonal to the congruence of lines form a confocal family of Dupin's cyclides. The reader will find attached to Maxwell's paper some good diagrams of the surfaces.

* One of these nodes is at the centre of inversion; if the original surface is a paraboloid or cylinder, the node is *biplanar* (a *binode*, Salmon, Art. 522); if the original surface is a parabolic cylinder, the node is *uniplanar* (a *unode*).

36. Quadratic line-complexes.

If l, m, n, l', m', n' denote the six Cayley-Plücker coordinates of a line*, the general quadratic complex is given by $B = 0$, where B is a quadratic form in the six variables l, m, n, l', m', n'. For the purpose of classification we note that the six variables are not independent but are connected by the identity $A = 0$, where

$$A = ll' + mm' + nn'.$$

Thus the classification will be made by the invariant-factors of the determinant $|B - \lambda A|$. The form A has zero signature, so that Klein's theorem (Art. 28) places no restriction on the types of invariant-factors; but, in order to avoid degenerate complexes, we must reject those cases where a factor of the determinant is also a factor of all the third minors†. On this account we must omit the following nine possibilities

[(1111) 11], [(1111) (11)], [(11111) 1], [(111111)],
[(2111) 1], [2 (1111)], [(21111)],
[(2211)],
[(3111)].

No singular cases will occur, since the determinant of A is -1; and thus, of the 66 cases mentioned in Art. 23, we must omit the 9 above and the 8 singular cases, leaving 49 cases, as found by Klein, Weiler and Segre. If the question of reality is considered the number of cases will be larger.

For fuller details the reader should consult Chapter XI of Jessop's *Line Complex* and the original papers quoted there. However two familiar complexes may be mentioned:—

(i) *the complex of normals to a set of confocal quadrics* (a special case of the tetrahedral complex), [(11) (11) (11)],

(ii) *the complex of tangents to a quadric*, [(111) (111)].

* The equations to the line can then be written (in Cartesians)

$$l' = ny - mz,\ m' = lz - nx,\ n' = mx - ly.$$

† *E.g.* [(1111) 11] gives

$$B = c_1 (k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2 + k_4 x_4^2) + c_5 k_5 x_5^2 + c_6 k_6 x_6^2,$$
$$A = k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2 + k_4 x_4^2 + k_5 x_5^2 + k_6 x_6^2,$$

and so the equations $B = 0$, $A = 0$ lead to

$$(c_5 - c_1) k_5 x_5^2 + (c_6 - c_1) k_6 x_6^2 = 0,$$

which degenerates into two linear complexes.

If $B=0$ gives the tangents to a quadric, it is not difficult to see that a line will be its own polar with respect to the quadric (and therefore a generator) when the six equations such as $\frac{\partial}{\partial l}(B-\lambda A)=0$ are satisfied. Now, since any root of $|B-\lambda A|$ is also a root of all its second minors*, it follows that these six equations reduce to four independent equations: these give two values for λ and then three of the coordinates of a generator can be expressed linearly in terms of the other three. Thus, for the central quadric

$$(a, b, c, f, g, h \between x, y, z)^2 + d = 0,$$

we find $B=(al^2+2hlm+\ldots)\,d+(bc-f^2)\,l'^2+2\,(fg-ch)\,l'm'+\ldots$.

Hence the conditions† for a generator are

$$\lambda^2 = \begin{vmatrix} a, & h, & g, & 0 \\ h, & b, & f, & 0 \\ g, & f, & c, & 0 \\ 0, & 0, & 0, & d \end{vmatrix}, \quad \lambda l' = d\,(al+hm+gn), \text{ etc.}$$

For the non-central quadrics it does not seem to be easy to find a simple symmetrical form for the result.

37. Application to the dynamical problem of small oscillations of a system with n degrees of freedom about a position of equilibrium.

This problem is known to depend on the consideration of two quadratic forms‡, the kinetic energy and the potential energy (see Routh, *Rigid Dynamics*, vol. I, Chap. IX, vol. 2, Chap. II; Whittaker, *Analytical Dynamics*, Chap. VII). The fact that the n velocities (instead of the n coordinates) appear in the kinetic energy will not affect the algebraical problem, as they are transformed by the same substitution as the n coordinates.

So consider two quadratic forms V, the potential energy, and T, the kinetic energy, but with the n coordinates in the place of the corresponding velocities. Then it is easy to see that the equation

$$\lambda T - V = 0$$

* Because $|B-\lambda A|$ has the characteristic $[(111)\,(111)]$; which means that if $(\lambda-c)^3$ is a factor of the determinant, then $(\lambda-c)^2$ is a factor of every first minor, and $(\lambda-c)$ of every second minor. It shews also that there are only two distinct roots for λ.

† St John's College Examination, Cambridge, 1897; Mathematical Tripos, 1898.

‡ Of course we make the usual assumptions, that the kinetic energy is calculated in the position of equilibrium; and that the potential energy is expansible near that position. Then V is not the actual potential energy, but merely the quadratic terms in the expansion of the potential energy.

is the same as Lagrange's determinantal equation for the periods of the system, where $\lambda = p^2$ and the period is $2\pi/p$. But, since T denotes the kinetic energy (modified by the substitution of coordinates for velocities), it is clear that the coefficients in the two forms T, V are real; and that T cannot vanish for any real values of the coordinates (other than zero).

Consequently the results of Art. 26 shew first that *all the values of λ are real*; and secondly that *all the invariant-factors of the determinant $|\lambda T - V|$ are linear.* It does not, of course, follow as yet that all the values of p are real, since some of the values of λ may be negative.

Now, if $c_1, c_2, \ldots, c_n$ denote the roots in λ, in virtue of Art. 26 the quadratic forms may be reduced by a real substitution to the types

$$\left.\begin{aligned} T &= a_1 y_1^2 + a_2 y_2^2 + \ldots + a_n y_n^2 \\ V &= a_1 c_1 y_1^2 + a_2 c_2 y_2^2 + \ldots + a_n c_n y_n^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1),$$

where the coefficients $a_1, a_2, \ldots, a_n$ are all *positive*, because T is a positive form. For the periods to be real, $c_1, c_2, \ldots, c_n$ must be positive and thus, *in order that the periods of the system may be real, it is necessary and sufficient that the form V may be positive, as well as T*; or, as the condition is usually stated, *the potential energy must be a minimum for stable equilibrium.* If, as occasionally happens*, V is positive but semi-definite, no oscillation will take place in certain of the coordinates.

It must not be forgotten that, if some of the c's are negative, the approximate equations which are used for the calculation of the periods will generally cease to be true after a certain time. For it is assumed in making the approximations that the coordinates remain *small* throughout the motion. But if $c_1 = -1$ say, the approximate equation of motion for the coordinate y_1 is

$$\frac{d^2 y_1}{dt^2} - y_1 = 0$$

which gives

$$y_1 = A \cosh t + B \sinh t\ ;$$

so that, unless the disturbance is of such a character that $A + B = 0$, the coordinate y_1 will increase as the time t increases, and consequently the primary assumption ceases to be valid after a certain time has elapsed, no matter how small the original disturbance may be. In such

* For example consider a sphere rolling inside a horizontal cylinder; if the sphere is displaced parallel to the generators, there will be no oscillation.

cases a different method of treatment has to be applied, if we wish to determine an approximation to the complete motion of the system.

When T, V are reduced to the types in (1) above, the coordinates $y_1, y_2, \ldots, y_n$ then appearing in them are called *normal or principal coordinates*; from our investigations of Chapter III it is clear that, in case all the roots of $|\lambda T - V| = 0$ are different, then T and V can be reduced simultaneously to sums of squares by *real* transformations (since the roots of $|\lambda T - V| = 0$ are real). But if these roots are not all different the types of Chapter III shew that it would apparently be impossible to make the reduction to sums of squares unless some further conditions are satisfied. However Weierstrass has proved that this reduction is always possible in the dynamical case; and, in fact, if we examine the reduced forms obtained in Chapter III, it is at once obvious that a positive definite form cannot be brought to any of these types by real transformations*, except to those corresponding to *linear* invariant-factors. When the invariant-factors are linear, the two forms can always be reduced simultaneously to sums of squares; and we have thus obtained a new proof of the second part of Art. 26.

In consequence of this result it follows that the expressions for the coordinates in terms of the time, deduced from Lagrange's equations, are simply periodic functions of the time, even when two or more of the periods are equal.

A simple example will make the theorem clearer. If we have two coordinates x, y and let $T = 2xy$ and $V = 2p^2xy + y^2$, we find the equations of motion

$$\frac{d^2x}{dt^2} = -p^2x, \quad \frac{d^2y}{dt^2} = -p^2y - x,$$

of which a special solution is

$$x = A \sin pt, \quad y = B \sin pt + \tfrac{1}{2}(At/p) \cos pt.$$

On the other hand, if $T = x^2 + y^2$ and $V = p^2(x^2 + y^2)$, we have

$$\frac{d^2x}{dt^2} = -p^2x, \quad \frac{d^2y}{dt^2} = -p^2y,$$

and then $x = A \sin pt$, $y = B \sin pt$ are special solutions.

Weierstrass's theorem amounts to saying that the first case is dynamically impossible; and that when two equal periods exist, the equations of motion are always reducible to the second form.

Lagrange had supposed that expressions of the type

$$(At + B) \sin (pt + a)$$

* The transformations would be real because complex values can only arise in the reduction if the determinantal equation has complex roots.

would appear in case two of the periods were equal; Weierstrass seems to have discovered the correct result for the first time in 1858 (see §§ 4, 5 of his first paper quoted in Art. 38). Of course, if terms of the type $(At+B)\sin(pt+\alpha)$ did occur in the coordinates, the primary assumption that all the coordinates are *small*, would cease to be true after a certain time; and a fresh method of investigation would be necessary, although one might anticipate instability.

Weierstrass's theorem has been rediscovered by several authors, of whom the chief are:—

SOMOFF, *Mém. de l'Acad. de St Pétersbourg*, 7e ser. t. 1, 1859, No. 14.
VILLARCEAU, *Comptes Rendus*, t. 71, 1870, p. 762.
JORDAN, *Comptes Rendus*, t. 73, 1871, p. 787; t. 74, 1872, p. 1395.
ROUTH, *Adams Prize Essay*, 1877, "*The Stability of Motion.*"

From a dynamical standpoint the method of proof given by Villarceau (for two coordinates) and Jordan (for the general case) is interesting; it is reproduced in Whittaker's *Analytical Dynamics*, Art. 77. This may be regarded, in a certain sense, as the dynamical version of Kronecker's proof (given in Art. 26 above). Of course all the proofs depend ultimately on the fact that $|\lambda T - V|$ has only linear invariant-factors, although this is not always mentioned explicitly in the work.

The reader may be tempted to suppose that Weierstrass's theorem can be extended to the case of oscillations about a state of steady motion; this is not the case in general, although it is true with certain restrictions on the type of motion*. To shew that the theorem is not true without restriction we may refer to the case of the sleeping top, for which the approximate equations of motion are known† to be

$$\ddot{x} + p\dot{y} - qx = 0,$$
$$\ddot{y} - p\dot{x} - qy = 0,$$

where p, q are positive constants.

If $2\pi/c$ is a period these give

$$\begin{vmatrix} c^2+q, & pc \\ pc, & c^2+q \end{vmatrix} = 0,$$

or
$$(c^2+q)^2 - p^2c^2 = 0.$$

* Weierstrass, *Berliner Monatsberichte*, 1879, p. 430; the investigation is reproduced in Whittaker's *Analytical Dynamics*, Art. 84. An alternative method will be found in the *Proc. Lond. Math. Soc.*, 1900, vol. XXXII, p. 79, § 2.

† Whittaker, *l.c.*, p. 202; Routh, *Rigid Dynamics*, vol. I, Art. 268.

This reduces to $(c^2-q)^2$ if $p^2=4q$, but the determinant has *not* then *four* invariant-factors of the types $(c \pm q^{\frac{1}{2}})$, but *two* of the types $(c \pm q^{\frac{1}{2}})^2$. In fact, it is not difficult to verify that, when $q=\frac{1}{4}p^2$, a particular solution of the equations is

$$x = At\cos(\tfrac{1}{2}pt), \quad y = At\sin(\tfrac{1}{2}pt),$$

which indicates that the method of approximation fails here after a certain time. It must not, however, be supposed that the system is then unstable; as a matter of fact it is stable, as was first proved by Klein*. One might perhaps be tempted to infer from Klein's result that in problems of steady motion linear invariant-factors are not necessary for stability; but an examination of the corresponding problem of a solid ring moving forward through a frictionless liquid, with circulation through the ring, shews that such cases may be either stable or unstable†. Thus the only safe inference to be drawn, when the invariant-factors are not linear, is that the method of approximation adopted is insufficient to settle the question of stability.

Example. As an illustration of Weierstrass's theorem consider a compound pendulum‡, formed of masses 1, m, suspended by strings of length 1, l. If the strings make small angles θ, ϕ with the vertical, we find

$$2T = \theta^2 + m(\theta + l\phi)^2, \quad 2V = g[\theta^2 + m(\theta^2 + l\phi^2)],$$

where T contains coordinates in place of velocities, as already explained.

The determinantal equation is

$$\begin{vmatrix} \lambda(1+m) - g(1+m), & \lambda lm \\ \lambda lm, & \lambda l^2 m - lmg \end{vmatrix} = 0,$$

or
$$\lambda^2 l^2 m - \lambda g(1+m)\, lm(1+l) + lm(1+m)\, g^2 = 0.$$

This has equal roots if

$$l^2 m^2 (1+m)\left[(1+m)(1+l)^2 - 4l\right] = 0.$$

It would therefore be possible to satisfy the condition for equal roots by taking

$$1 + m = 4l/(1+l)^2,$$

or
$$m = -(1-l)^2/(1+l)^2;$$

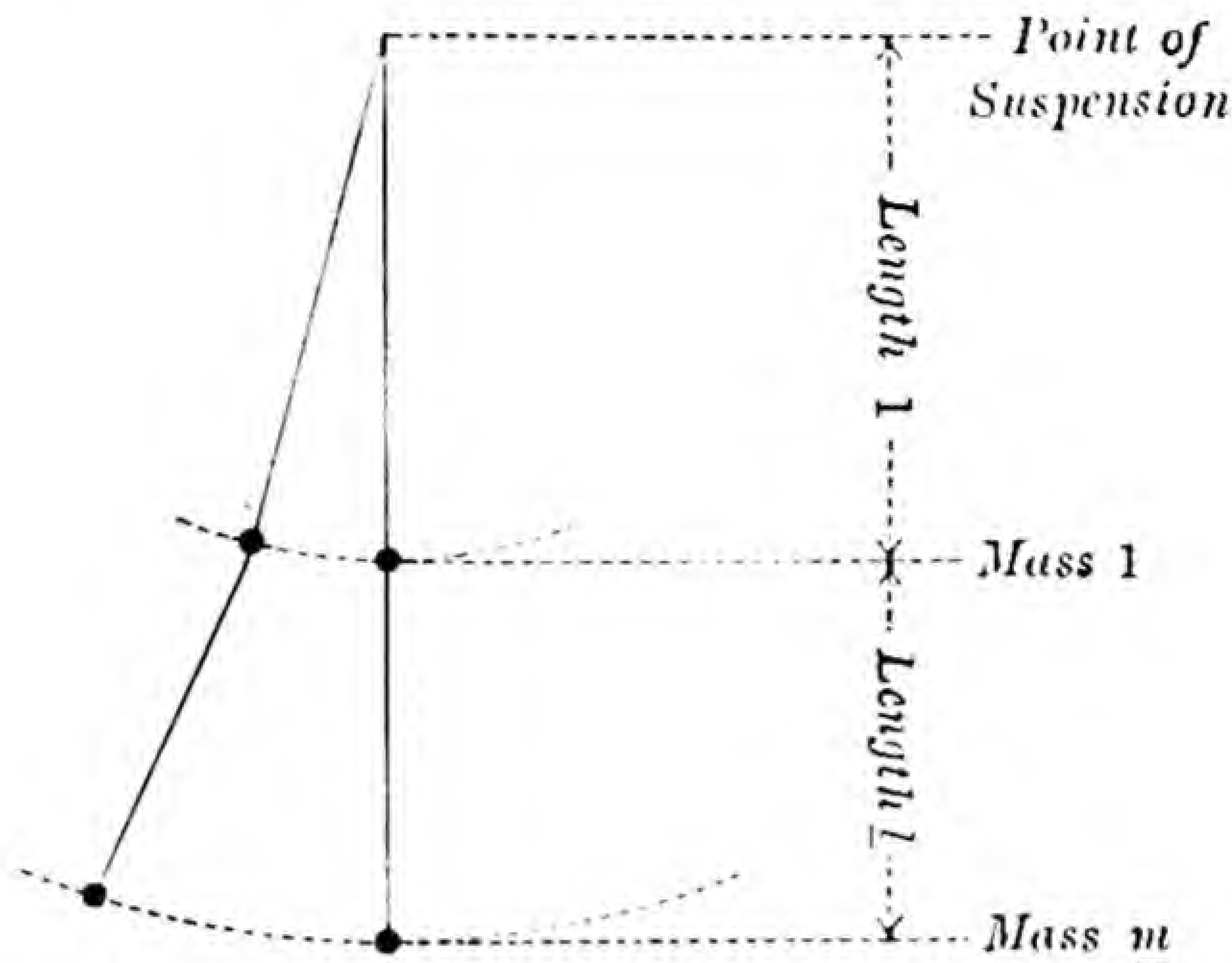
and then it will be seen that the determinant has *not* linear invariant-factors.

* *Bulletin Amer. Math. Soc.* Ser. 2, vol. III, 1897, pp. 129, 292; previous workers, following Routh (*Adams Prize Essay*, see also *Rigid Dynamics*, vol. I, Art. 268), had supposed the case $q=\frac{1}{4}p^2$ to be unstable.

† *Proc. Lond. Math. Soc.*, Ser. 1, vol. XXXIII, 1901, p. 326 (see p. 340 for the special result quoted).

‡ Stokes (*Math. and Phys. Papers*, vol. IV, p. 334; vol. V, p. 354) has considered this case from a different point of view. At that time he was apparently unacquainted with Weierstrass's work.

But this does not invalidate Weierstrass's general theorem; for we have introduced a *negative* mass m. This algebraic possibility is tacitly excluded by the assumption that the kinetic energy must be positive.



If we restrict ourselves to *positive* masses, the condition for equal roots

$$(1+m)(1+l)^2-4l=0,$$

or

$$(1-l)^2+m(1+l)^2=0,$$

gives at once $m=0$, $l=1$.

The determinant then is identically zero but has the one invariant-factor $(\lambda-g)$.

38. References.

For the sake of reference, we add a brief list of papers, which are more or less connected with the theory of invariant-factors and their applications to the classification of quadratic forms.

It may be useful to point out that some of the papers, notably the second one by Weierstrass, refer in the first instance to families of bilinear forms of the type

$$\Sigma(\lambda a_{rs}-b_{rs})\,x_r y_s,$$

in which a_{rs} is not usually equal to a_{sr} nor b_{rs} to b_{sr}.

It will probably be thought that the results of such investigations will include the theory of quadratic forms as a special case; and to a certain extent this is true. In investigations such as Weierstrass's, it is of fundamental importance to establish the existence of a sequence of ***regular*** minors (Art. 25) each derived from the preceding

by striking out a row and column*. Now, as was pointed out in Art. 25, it is not always possible to find *symmetrical* regular minors in the determinant of a family of quadratic forms; and this will be found to import a difficulty into the theory which is not encountered in dealing with bilinear forms.

The difficulty is partially avoided in Darboux's paper by using bordered determinants instead of minors; part of this paper is translated in Chapter XI of Scott's *Theory of Determinants*; and more of it will be found in Chapter XI of Jessop's *Line Complex*.

The only existing text-book is Muth's *Elementartheiler* (Leipzig, 1899), to which reference has already been made. A review of this book, together with a short historical account will be found in the *Bulletin*† of the American Mathematical Society, April, 1901.

Fuller references are given in W. F. Meyer's report on Invariant-theory in the *Jahresbericht der Deutschen Math. Verein.* Bd. 1, pp. 106—118, and in the *Encyklopädie der Mathematischen Wissenschaften*, Bd. 1, pp. 327—334; and some additions and corrections are given in the *Archiv für Math. und Physik*, (3) Bd. 2, p. 359.

A number of articles dealing with various points in the theory have been published by the author. They will be found in the *Proc. Lond. Math. Soc.*, 1900—1902, vols. XXXI—XXXIII; *Proc. Camb. Phil. Soc.* 1900—1902, vols. X, XI; and in the *American Journal of Mathematics*, 1901, vol. XXIII, p. 235. A summary of results, from which this tract has been enlarged, will be found in the *Quarterly Journal of Mathematics*, vol. XXXIII, 1901, p. 85.

The most notable papers on the transformation and reduction of forms are the following:—

G. DARBOUX, "Mémoire sur la théorie algébrique des formes quadratiques," *Journal de Math.* (Liouville), t. 19, 2me Série, 1874, p. 347.

C. JORDAN, "Mémoire sur la réduction et la transformation des systèmes quadratiques‡," *Journal de Math.* (Liouville), t. 19, 2me Série, 1874, p. 397.

* The proof of this theorem is one of the chief difficulties in developing the theory of reduction by Weierstrass's or Darboux's methods; and it is partly on this account that we have preferred to exhibit the theory according to Kronecker's methods. The theorem was first given by H. J. S. Smith (in his first paper quoted below) in reference to determinants of *integers*; and it was extended by Frobenius to determinants whose elements are polynomials in λ.

† Second Series, vol. VII, pp. 308—316.

‡ Some of the results given in this paper require correction, as remarked by Kronecker, and afterwards admitted by Jordan.

L. Kronecker, "Ueber Schaaren quadratischer Formen*," *Berliner Monatsberichte*, 1868, p. 339, and *Ges. Werke*, Bd. 1, p. 165.

—— "Ueber Schaaren von quadratischen und bilinearen Formen†," *Berliner Monatsberichte*, 1874, pp. 59, 149, 206; and *Ges. Werke*, Bd. 1, pp. 351, 373, 382.

—— "Algebraische Reduktion der Schaaren quadratischer Formen," *Berliner Sitzungsberichte*, 1890, p. 1375, and 1891, pp. 9, 33.

L. Stickelberger, "Ueber Schaaren von bilinearen und quadratischen Formen," *Journal für Math.* (Crelle), Bd. 86, 1879, p. 20.

J. J. Sylvester's papers are quoted above in Arts. 5, 9, 10, 11, 23, 26, and 28.

K. Weierstrass, "Ueber ein die homogenen Function zweiten Grades betreffendes Theorem, nebst Anwendungen auf die Theorie der kleinen Schwingungen‡," *Berliner Monatsberichte*, 1858, p. 207, and *Ges. Werke*, Bd. 1, p. 233.

—— "Zur Theorie der bilinearen und quadratischen Formen§," *Berliner Monatsberichte*, 1868, p. 310, and *Ges. Werke*, Bd. 2, p. 19.

On the theory of invariant-factors and regular minors may be mentioned :—

H. J. S. Smith, "On Systems of Linear Indeterminate Equations and Congruences," *Philosophical Transactions*, vol. CLI, 1861, p. 318; and *Coll. Math. Papers*, vol. I, p. 367.

—— "On the Arithmetical Invariants of a Rectangular Matrix of which the Constituents are Integral Numbers," *Proceedings of the London Mathematical Society*, 1873, vol. IV, p. 237; and *Coll. Math. Papers*, vol. II, p. 67.

G. Frobenius, "Theorie der linearen Formen mit ganzen Coefficienten," *Journal für Math.*, Bd. 88, 1880, p. 116.

—— "Ueber die Elementartheiler der Determinanten," *Berliner Sitzungsberichte*, 1894, p. 31.

K. Hensel, "Ueber reguläre Determinanten," *Journal für Math.*, Bd. 114, 1894, p. 25.

—— "Ueber die Elementarteiler componirter Systeme," *ibid.*, p. 109.

* Contains the substance of Art. 26.

† Contains the methods of Arts. 13, 17, 20.

‡ Contains the theorems of Art. 26 proved in a different manner.

§ Contains the first systematic reduction of a family of forms.

APPENDIX.

The geometrical interpretation of the transformations in Art. 7.

In the first place, we note that if a quadratic form A contains three variables, x_1, x_2, x_3, and if one of these (say x_1) appears only as a square x_1^2, then the side x_1 is the polar of the opposite vertex $\mathbf{X}_1$ of the triangle of reference, with respect to the conic $A = 0$.

On the other hand, if A contains x_1 and x_2 only in the product x_1x_2, the sides x_1, x_2 will touch the conic $A = 0$ at two vertices of the triangle of reference.

Corresponding results are true for a form in four variables if we read *plane, quadric, tetrahedron* for *side, conic, triangle*, respectively.

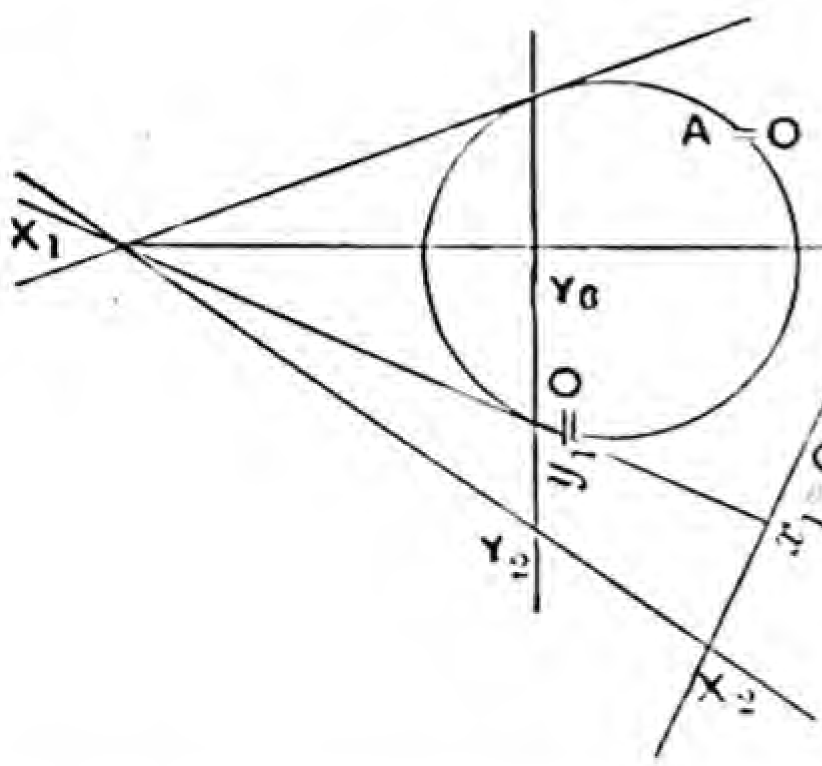
Bearing these facts in mind, it is easy to interpret geometrically the four transformations of Art. 7; it will be sufficient to give the details for three variables, leaving the reader to make the changes necessary when the work is applied to four variables.

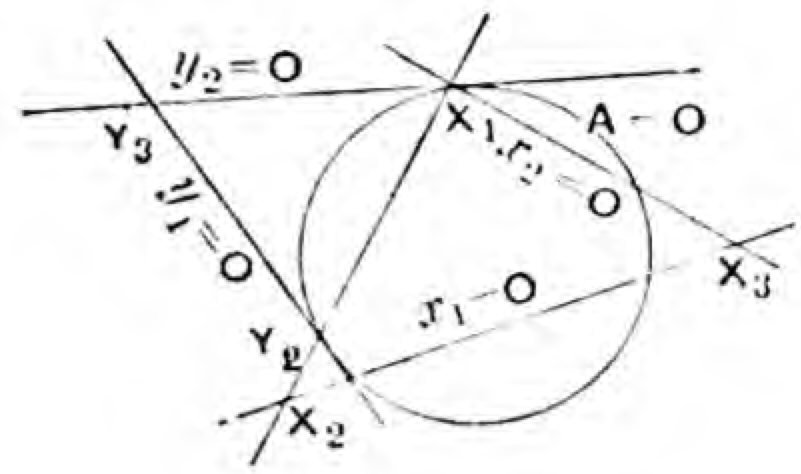
If it is convenient to *retain the vertex* $\mathbf{X}_1$ *of the fundamental triangle unaltered*, while the conic A is to be simplified, we naturally find first the polar of $\mathbf{X}_1$ and use this line as a side of the new triangle of reference; there are then two alternatives, according as $\mathbf{X}_1$ is not or is a point on the conic :—

(i) *If* $\mathbf{X}_1$ *is not on the conic*, the polar will be taken as y_1, the side opposite to $\mathbf{X}_1$ in the new triangle; and in the actual transformation of Art. 7, $\mathbf{Y}_2$, $\mathbf{Y}_3$ are taken on the sides x_3, x_2 of the original triangle.

(ii) *But if* $\mathbf{X}_1$ *is on the conic*, the polar passes through $\mathbf{X}_1$, and may be taken as y_2, while y_1 is any other tangent to the conic. In the

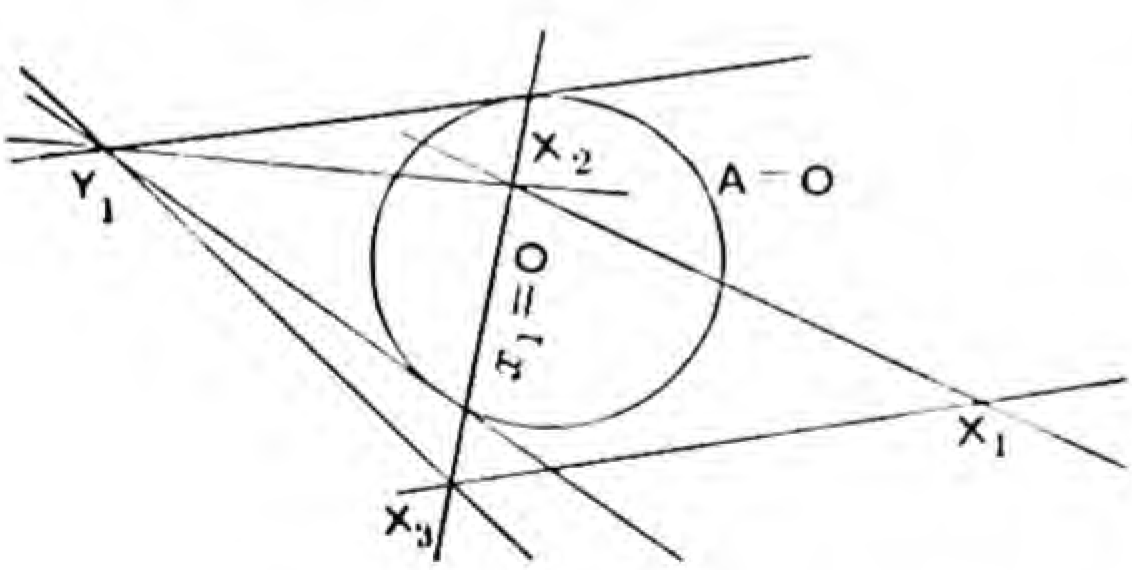
text, it is supposed that $\mathbf{X}_2$ is not on the polar y_2; the line $\mathbf{X}_1\mathbf{X}_2$ then cuts the conic again in $\mathbf{Y}_2$, and y_1 is the tangent there.

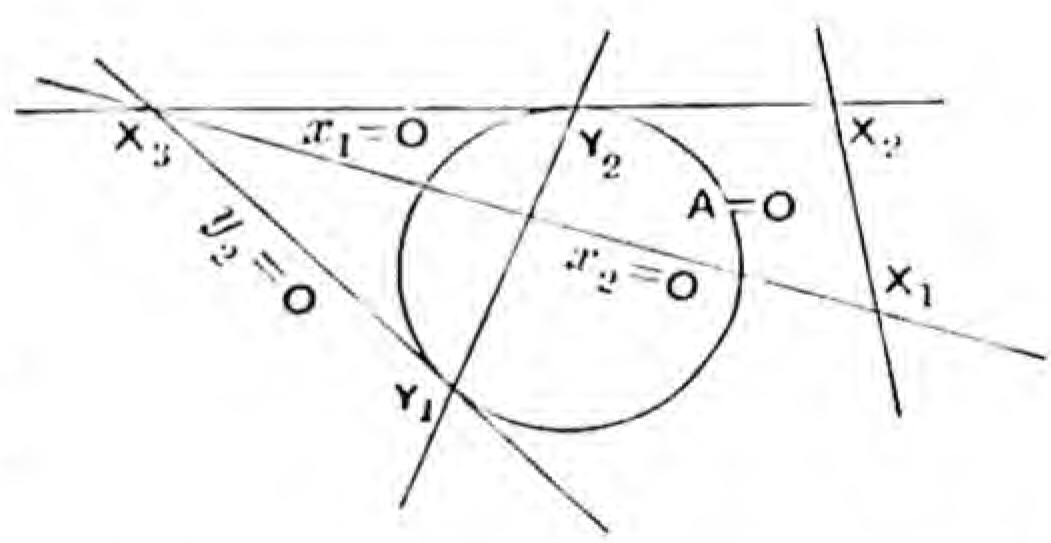




On the other hand, it may be better to *keep the side x_1 of the fundamental triangle unaltered*; then we begin by taking the pole of x_1 as a new vertex of the triangle of reference. The distinction now turns on whether x_1 is not or is a tangent to the conic:—

(iii) *If x_1 does not touch the conic*, the pole can be taken as $\mathbf{Y}_1$, the vertex opposite to x_1; and in the text of Art. 7, the vertices $\mathbf{X}_2$, $\mathbf{X}_3$ are left unchanged.





(iv) *But if x_1 touches the conic*, the pole is on x_1 and may be taken as $\mathbf{Y}_2$, while $\mathbf{Y}_1$ is any other point on the conic. In the text, it is supposed that x_2 does not pass through $\mathbf{Y}_2$; then $\mathbf{Y}_1$ is the point of contact of the other tangent (y_2) which passes through the intersection of x_1, x_2.

From these explanations it is clear that (iii), (iv) are transformations reciprocal to (i), (ii) respectively; and further that (i) and (ii) are alternatives, one or other being always a possible method; similarly (iii) and (iv) are alternatives.

From the algebraical point of view, the transformations (i) and (ii) usually involve less labour than (iii) and (iv); and, as a matter of

fact (iv) is hardly ever needed in practical work (compare Art. 17, p. 42). However, the use of (iv) enables us to abbreviate considerably the *description* of the general process[1] given in Art. 20, p. 52; and for this reason, as well as for completeness, it is advantageous to explain this method as well as the other three.

[1] As explained in the small type on that page, the general process will often be found to imply unnecessary labour in numerical work; and this extra trouble is due to the application of method (iv).

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